A CLASS OF PRETZEL KNOTS

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1. **Introduction.** Let $K \subset S^3$ be a tame knot with group $G = \pi_1(S^3 - K)$ and Alexander polynomial $\Delta(t)$. By a *surface spanning* K is meant a tame, connected, orientable 2-manifold with boundary K. The minimum of the genera of all surfaces spanning K is the *genus* g of K.

Among all possible knot groups those for which the commutator subgroup $[G]$ is finitely generated are of special interest. Suppose S is a surface of minimal genus that spans K. The inclusion $i : S^3 - S \rightarrow S^3 - K$ induces a homomorphism of the fundamental groups which maps $\pi_1(S^3 - S)$ into $[G]$. Neuwirth has shown [3] that $[G]$ is finitely generated iff $i : \pi_1(S^3 - S) \rightarrow [G]$ is an isomorphism and both groups are free of rank $2g$. Hence for $[G]$ to be finitely generated it is obviously necessary that

(i) $\pi_1(S^3 - S)$ *is free of rank* $2g$.

According to Theorem 1 of [4], if $[G]$ is free of rank $2g$, then

(ii) $| \Delta(0) | = 1$ *and* degree $\Delta(t) = 2g$.

(It is not hard to show that the induced mapping $i : H_1(S^3 - S) \rightarrow [G]/[[G]]$ is an isomorphism iff (ii) holds.) We conjectured that (i) and (ii) were sufficient for $[G]$ to be free. (Here and elsewhere in this paper a "free" group is assumed to be finitely generated.) Recently Murasugi showed that the conjecture is true for alternating knots. (In this case (i) and (ii) are implied by $| \Delta(0) | = 1$.) We show in this paper that the conjecture does not hold in general.

We consider a class of pretzel knots [5; 9] of genus ≤ 1, namely, projections like those in Figure 1 with an odd number of knots having crossings in each braid. Such a knot is specified by a triple of integers $(2p + 1, 2q + 1, 2r + 1)$ whose absolute values are the numbers of crossings in the braids and which are $\pm$ according as the twists are right or left handed. Triples which differ only by a permutation or change of sign throughout obviously specify equivalent knots. We shall see that to within these allowable changes, *only* $(1, 1, 1)$ *and* $(3, -1, 3)$ *specify the trefoil knot, and only* $(1, 1, -3)$ *specifies the figure-eight knot.*

Every knot K specified by $(2p + 1, 2q + 1, 2r + 1)$ will be shown to be spanned by a surface S of genus 1, which satisfies condition (i) unless K is trivial, and which has

$$V = \begin{pmatrix} p + q + 1 & -q - 1 \\ -q & q + r + 1 \end{pmatrix}$$

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