# MODULARY GROUPS OF t × t MATRICES

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**Introduction.** In this note we undertake an initial investigation of the modulary groups $\mathfrak{M}(a, b)$ of $t \times t$ matrices. The determination of the structure of these groups is reduced to the case when $a$ and $b$ are each powers of the same prime $p$, and under certain conditions these are completely determined. In addition we prove some lemmas on the principal congruence subgroups of the $t \times t$ modular group which are of interest in themselves. Finally we mention an open question that we have not been able to settle.

Let $\Gamma$ be the group of rational integral $t \times t$ matrices of determinant 1, $\Gamma(n)$ the principal congruence subgroup of $\Gamma$ of level $n$. $\Gamma(n)$ consists of all matrices $A \in \Gamma$ satisfying $A \equiv I \pmod{n}$. Let $m$, $n$ be positive integers and set

$$(m, n) = d, \qquad [m, n] = \delta.$$

Lemma 1. *Let $A \in \Gamma(d)$. Then $X$ can be determined so that*

$$X \equiv I \pmod{m}, \tag{1}$$

$$X \equiv A \pmod{n}, \tag{2}$$

$$\det X = 1. \tag{3}$$

*Proof.* Since $A \in \Gamma(d)$ we can write $A = I + dB$. Set $X = I + mY$. Then (1) is satisfied and (2) becomes

$$mY \equiv dB \pmod{n},$$

$$\frac{m}{d} Y \equiv B \left(\operatorname{mod} \frac{n}{d}\right).$$

Since $(m/d, n/d) = 1$, this has a solution $Y$. Hence there is an $X_0$ satisfying (1) and (2). Furthermore, $\det X_0 \equiv 1 \pmod{m}$, $\det X_0 \equiv \det A \equiv 1 \pmod{n}$, so that $\det X_0 \equiv 1 \pmod{\delta}$. Now determine $X$ so that $X \equiv X_0 \pmod{\delta}$, $\det X = 1$ (see [2; 374]). Then this $X$ satisfies the conditions of the lemma.

Lemma 1 now implies

Lemma 2. *Let $A \in \Gamma(d)$. Then $B$ and $C$ may be found such that $B \in \Gamma(m)$, $C \in \Gamma(n)$ and $A = BC$.*

*Proof.* Determine $X$ as in Lemma 1. Choose $B = X$, $C = X^{-1}A$.

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