INVARIANT SUBSPACES OF TRIDIAGONAL OPERATORS

By P. L. Duren

1. Introduction. The spectral resolution of a self-adjoint operator is essentially an analysis of the entire space as a direct sum of subspaces in each of which the operator is a constant multiple of the identity. A similar decomposition is available for normal operators, but not for linear operators in general. In recent years many investigations of non-normal operators have been directed toward more modest ends: (1) location and classification of the spectrum; and (2) description of the invariant subspaces.

One of the outstanding results of the second type is Beurling's characterization [2] of the invariant subspaces of the shift operator

$$S: (x_0, x_1, x_2, \cdots) \to (x_1, x_2, x_3, \cdots) \tag{1}$$

in the Hilbert space l_2 of complex square-summable sequences. An equivalent problem, Beurling observes, is to describe the invariant subspaces of the multiplication operator

$$M: F(w) \to wF(w) \tag{2}$$

in the space H_2 of analytic functions in the unit disk. This is accomplished through a canonical factorization $F(w) = F_0(w)F_1(w)$ of each function $F(w) \in H_2$ into the product of its *inner factor*

$$F_0(w) = B(w) \exp\left\{-\int_0^{2\pi} \frac{e^{it} + w}{e^{it} - w}\, d\mu(t)\right\}$$

and its *outer factor*

$$F_1(w) = e^{i\beta} \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log | F(e^{it}) | \frac{e^{it} + w}{e^{it} - w}\, dt\right\}.$$

Here $B(w)$ is a Blaschke product formed from the zeros of $F(w)$, and μ is a non-negative singular measure. (We implicitly assume $F(w) \not\equiv 0$.)

Beurling introduces a lattice structure into the set of inner functions by defining a notion of *divisibility* as follows. Let F_0, G_0 be inner functions constructed with zeros $\{a_n\}$, $\{b_n\}$ and measures μ, ν, respectively. Then F_0 is a divisor of G_0 if and only if $\{a_n\}$ is a subset of $\{b_n\}$ and the set function $\mu \leq \nu$. Any two inner functions F_0 and G_0 have, with obvious meanings, a *greatest common divisor* $F_0 \wedge G_0$ and a *least common multiple* $F_0 \vee G_0$.

For any fixed $F \in H_2$, the subspace $\mathcal{P}[F]$ spanned by the iterates $\{M^nF\}_0^\infty$ is certainly invariant under M. Beurling proves that *every* invariant subspace is of this type, and that the lattice of invariant subspaces of M is isomorphic

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