EMBEDDING A REGULAR NEIGHORHOOD OF THE SINGULAR LOCUS OF A 2 DIMENSIONAL POLYHEDRON IN E^3

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The purpose of this paper is to state combinatorial conditions which are necessary and sufficient for the existence of semi-linear embeddings of a regular neighborhood of a component of the singular locus of a 2 dimensional polyhedron in E^3.

It is known that a satisfactory solution of the general problem of embedding 2 dimensional polyhedra in E^3 would determine the correctness of the Poincaré conjecture that a compact simply connected 3 manifold is a 3 sphere, [1], [4]. The conditions for the theorem stated here are not elegant. On the other hand, one can use the methods to determine the embeddability of many 2 polyhedra without excessive work. My interest in this problem resulted from being in the vicinity of R. H. Bing, and talking with him.

1. Introduction. Let K be a 2 dimensional compact connected polyhedron, and L the singular locus of K (that is, L consists of all points of K which do not have neighborhoods homeomorphic with an open disk). The components $L_1, \cdots, L_n$ of L form a subpolyhedron in the 1 skeleton of any triangulation of K. Each such component L_i has regular neighborhoods M_i in K, (with respect to a triangulation t), and the boundary of M_i is a collection $\{S_{i\alpha}\}$ of disjoint 1 spheres. It is no restriction to assume that $\bar{M}_i \cap \bar{M}_j = \emptyset$, $i \neq j$. Each $\bar{M}_i$ is homeomorphic to the mapping cylinder C_f of a semi-linear map f: $\bigcup_\alpha S_{i\alpha} \to L_i$, where each $S_{i\alpha}$ is a 1 sphere, $S_{i\alpha} \cap S_{i\beta} = \emptyset$, $\alpha \neq \beta$, and L_i a 1 dimensional polyhedron, or a point. In what follows we shall discuss the embedding of mapping cylinders of the type described.

Thus, let $\{S_\alpha\}$ be a collection of disjoint 1 spheres, and denote $\bigcup_\alpha S_\alpha$ by S. Let Y be a 1 dimensional polyhedron and $f: S \to Y$ a semi-linear map. Recall that the mapping cylinder C_f is the set $\{(S \times I) \cup Y\}/\varphi$, where φ is the identification function which identifies $(x, 1) \in (S \times I)$ and $f(x) \in Y$. The set $(S \times 1)/\varphi$ is homeomorphic with the image of f. The natural embedding of Y in C_f is defined by $p(y) = \{y\}/\varphi$, and the natural embedding of S in C_f by $q(x) = \{(x, 0)\}/\varphi$. The mapping cylinder can be triangulated so that p and q are semi-linear embeddings.

Let y_0 be a vertex of Y. Then $p(y_0)$ is a vertex of C_f, and $p(Y)$ a subpolyhedron of C_f. Since Y (unembedded) is rarely mentioned, we ease the notation by writing y for $p(y)$. Thus the star of $p(y_0)$ in C_f with respect to a triangulation t of C_f will be denoted by (st $y_0 : C_f, t$) or briefly (st $y_0 : C_f$). The boundary of (st $y_0 : C_f$) is a 1 dimensional polyhedron, call it $K_f(y_0)$.

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