LOWER BOUNDS FOR THE MAXIMUM MODULI OF CERTAIN CLASSES OF TRIGONOMETRIC SUMS

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1. **Introduction.** Consider the trigonometric sum

$$S(k_1, \cdots, k_n; p) = \sum_{u=0}^{p-1} \exp\left[\frac{2\pi i}{p}(k_1 u + \cdots + k_n u^n)\right].$$

where the k_i's and n are arbitrary positive integers and p is any positive prime. Estimation of the absolute value of this sum has received considerable attention. A bound of $(n-1)p^{\frac{1}{2}}$ was conjectured by Hasse [2; 53] in 1935 and by Mordell [3; 67] in 1946, the latter pointing out that it should follow from Weil's proof [4] of the Riemann hypothesis on the zeta-function for algebraic function fields. In 1957 Carlitz and Uchiyama [1], using Weil's result, actually obtained the estimate of $(n-1)p^{\frac{1}{2}}$.

However, there is no information, other than numerical, about the sharpness of this upper bound when $n > 2$, and there is no non-trivial estimate of the behavior of this sum for the case where $n > p^{\frac{1}{2}} + 1$. In particular, there has been given no lower bound on

$$M(n, p) = \max | S(k_1, \cdots, k_n; p) |$$

where the maximum is taken over all $(k_1, \cdots, k_n) \not\equiv (0, \cdots, 0) \pmod{p}$. The present paper fills this gap by presenting a lower bound for $M(n, p)$ whenever p is a positive prime such that $p > 2$ and when $1 < n < p - 1$, i.e., in all non-trivial cases. This lower bound is exhibited in the following inequality:

$$M(n,p) > \left[(n!)^2\binom{p}{n} - p^n\right]^{\frac{1}{2n}}. \tag{1.1}$$

This bound is significant for two reasons. First, it throws much more light on the magnitude of $M(n, p)$ in the cases where the upper estimate $(n-1)p^{\frac{1}{2}}$ is non-trivial, i.e., in the cases where $n < p^{\frac{1}{2}} + 1$. Second, there is no other general assertion about the size of $M(n, p)$ if $n > p^{\frac{1}{2}} + 1$.

2. **Results.** Several definitions and lemmas are needed to establish our result.

DEFINITION 1.

$$S(k_1, \cdots, k_n; p) = \sum_{u=0}^{p-1} \exp\left[\frac{2\pi i}{p}(k_1 u + k_2 u^2 + \cdots + k_n u^n)\right]$$

where

$$0 \le k_r \le p - 1.$$

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