THE CESARI-CAVALIERI AREA OF A SURFACE

By R. E. Fullerton

1. **Introduction.** In the theory of continuous surfaces as undertaken by Cesari [1], Radó [6] and others, the notion of area of a surface is of fundamental importance. Thus if $\mathcal{S}$ is the class of all Fréchet surfaces, an area functional is a non-negative linear functional on $\mathcal{S}$ which satisfies certain conditions [1]. In particular, the classical Lebesgue definition of area defines this functional in terms of approximations of the given surface by polyhedral surfaces [1]. Other definitions make use of certain concepts of total variation for plane-to-plane mappings. Some of the most fundamental theorems in the theory of surfaces are those which prove the equivalence of certain of these definitions. The area functional discussed here is based upon the Cavalieri inequality of Cesari and depends upon a generalization of the ancient Cavalieri principle in which the area of a surface is expressed essentially in terms of an integral of the length function of a family of curves which cover the surface. Thus, from the nature of the definition, it appears possible that this notion of area may be of use in the treatment of problems involving integrals on a surface.

To provide the necessary background, the concept of a contour, generalized length, and the Cavalieri inequality are introduced in the next section and the Cesari–Cavalieri area functional $K(S)$ is defined. In §3 it is shown that the functional $K(S)$ coincides with the elementary area of a polyhedral surface. §4 contains proofs that the area $K(S)$ has the properties listed by Cesari [1] and hence defines an acceptable area functional. In the final sections it is shown that $K(S)$ coincides with the Lebesgue area $L(S)$ for a smooth non-parametric surface. It appears probable that $K(S) = L(S)$ for a general continuous surface but to date, this remains an open question.

2. **Definitions and preliminary results.** Let Q be a two-dimensional compact manifold, with or without boundary. Let $T\colon Q \to E_N$ be a continuous mapping of Q into N dimensional Euclidean space. The mapping T thus defines a Fréchet surface S, [1]. Let $[S]$ be the set of points in E_N occupied by the surface and let $f\colon [S] \to$ Reals be a real-valued continuous function defined over $[S]$. Let $C(t)$, $D^-(t)$ denote respectively the set of points $p \in Q$ for which $f(T(p)) = t$, $f(T(p)) < t$. The set $C(t)$ is called the *contour* corresponding to t for f and T and the set $[D^-(t)]^*\backslash D^-(t)$ is the *lower border* of the contour $C(t)$. As in [3], the length of the image of the lower border of the contour can be defined by covering each component of the lower border by a finite family of coordinate neighborhoods, by defining in each such neighborhood a generalized

Received March 15, 1962. This research was done while the author was a fellow of the John Simon Guggenheim Memorial Foundation.