CONVOLUTION OPERATORS AND L(p, q) SPACES

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Introduction. It is easily seen that for the convolution of two functions,

$$(f * g)(x) = \int f(x - t) g(t)\, dt$$

we have

$$\| f * g \|_1 \leq \| f \|_1 \; \| g \|_1 ,$$

$$\| f * g \|_\infty \leq \| f \|_\infty \; \| g \|_1 ,$$

$$\| f * g \|_\infty \leq \| f \|_1 \; \| g \|_\infty .$$

Professor E. M. Stein suggested that the above three inequalities were all that were needed in order to develop many of the classical theorems on convolutions. Thus the notion of convolution operator as defined in §I is due to him. In fact, a few days after the author first proved the principal theorem of this paper (Theorem 2.6) for ordinary convolution on Euclidean n-space, Stein proved the theorem for a convolution operator by means of an elegant, but high-powered method. It was then that the elementary method of attack described in this paper was discovered.

In 1950 G. G. Lorentz [2] first discussed the spaces (in the notation of the present article) $L(p, 1)$ and $L(p, \infty)$. He also defined (using rather different notation) the spaces $L(p, q)$ for $q \leq p$.

Recently A. P. Calderón introduced the present notation and described the basic properties of these spaces. We have needed some of these properties. (Lemma 2.2 and Lemma 2.5).

The reader who wishes to skim this paper should familiarize himself with the notation at the beginning of §I (through Equation 1.3), and then proceed to the statements of Lemma 1.5 and Theorem 1.7.

In §II, he should read the introductory material through the statement of Lemma 2.2, Lemma 2.5, and Theorem 2.6 which is the principal result of this paper. He may then proceed to any remarks of §III he finds interesting.

I. **Convolution operators.** *Definition* 1.1. *Given three measure spaces* (X, μ), $(\bar{X}, \bar{\mu})$, *and* (Y, ν), *a bilinear operator*, T, *which maps measurable functions on* X *and* $\bar{X}$ *into measurable functions on* Y *is called a convolution operator if*

1. $T(f_1 + f_2 , g) = T(f_1 , g) + T(f_2 , g)$,
2. $T(f, g_1 + g_2) = T(f, g_1) + T(f, g_2)$,

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