THE STABILITY OF FINITE DIFFERENCE APPROXIMATIONS TO SECOND ORDER LINEAR PARABOLIC DIFFERENTIAL EQUATIONS

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1. Introduction. We consider explicit finite difference approximations to the initial value problem for a second order linear parabolic differential equation with variable coefficients in two independent variables:

$$a(x, t) \frac{\partial^2 u}{\partial x^2} + b(x, t) \frac{\partial u}{\partial x} + c(x, t)u - \frac{\partial u}{\partial t} = f(x, t). \tag{1.1}$$

Fritz John [4] has studied finite difference approximations to (1.1) of the form

$$u(x, t + \tau) = \sum_r d^r(x, t, h)u(x + rh, t) - \tau f(x, t), \tag{1.2}$$

and has given sufficient conditions for the stability and smoothness of solutions of the initial value problem for (1.2). John has also investigated the convergence of solutions of the initial value problem for (1.2) to solutions of the corresponding problem for (1.1) as well as the existence of generalized solutions of (1.1), but we shall not concern ourselves with these questions here. One of the main tools in John's work is an estimate for the fundamental solution of a difference equation of the form (1.2) with $f \equiv 0$ and the coefficients d^r independent of x. Our principal result in this note is a new estimate for this fundamental solution and its differences. Our estimate (Theorem I in §2 below) is sharper than John's and reduces to known estimates for the fundamental solution of

$$a(t) \frac{\partial^2 u}{\partial x^2} + b(t) \frac{\partial u}{\partial x} + c(t)u - \frac{\partial u}{\partial t} = 0$$

when $h, \tau \to 0$. The latter is not only an aesthetic advantage, but also a technical advantage since it permits us to obtain the finite difference analogues of various known results for parabolic differential equations.

In [4] John uses his estimate for the fundamental solution of the homogeneous difference equation with coefficients independent of x to derive, via the parametrix method, *a priori* estimates for the solution of the initial value problem for (1.2) and for its first difference with respect to x (i.e., stability and smoothness of the solution). We apply our estimate in §3 to obtain John's stability and smoothness results under reduced regularity assumptions on the coefficients of (1.2). The version of the parametrix method which we employ in §3 is based on our proof of the uniqueness of the solution of the initial value problem for

Received March 26, 1962. This work was supported in part by the Office of Naval Research under Contract Nonr-710(16), (NR 043 041), and in part by the National Science Foundation under a Post-Doctoral Fellowship.