INFINITE SERIES AND NONNEGATIVE VALUED INTERVAL FUNCTIONS

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1. **Introduction.** In this paper we extend a previous result of the author [1] and demonstrate (Theorem 2) that if H is a real nonnegative-valued function of subintervals of the number interval $[a, b]$, then the integral (§3)

$$\int_{[a,b]} H(I)$$

exists if and only if for each real-valued nondecreasing function m on $[a, b]$ there is a number p such that $0 < p < 1$ and the integral

$$\int_{[a,b]} [H(I)]^p[dm]^{1-p}$$

exists.

2. **A lemma concerning infinite series.** In this section we prove a preliminary lemma about infinite series with nonnegative-valued terms.

Lemma S. *If $\{a_k\}_{k=1}^{\infty}$ is a sequence of nonnegative numbers whose sum diverges, then there is a sequence $\{e_k\}_{k=1}^{\infty}$ of nonnegative numbers whose sum converges such that $\sum a_k^p e_k^{1-p}$ diverges for all p in $(0, 1)$.*

Proof. If v is in $(0, 1)$, then by the Banach–Steinhaus theorem there is a sequence $\{c_k\}_{k=1}^{\infty}$ of nonnegative numbers such that $\sum c_k \leq 1$ and $\sum a_k^v c_k^{1-v} = \infty$. For each positive integer k we let $b_k = \min\{a_k, c_k\}$, so that $b_k \leq a_k$ and $\sum b_k \leq \sum c_k \leq 1$. Considering the set of all j such that $c_j \leq a_j$ and the set of all j' such that $a_{j'} < c_{j'}$, we see that $\sum a_k^v b_k^{1-v} = \sum a_j^v c_j^{1-v} + \sum a_{j'} \geq \sum a_k^v c_k^{1-v} - \sum a_{j'}^v c_{j'}^{1-v} \geq \sum a_k^v c_k^{1-v} - \sum c_{j'} = \infty$.

Therefore for each positive integer $q > 1$ there is a sequence $\{b_k^{(q)}\}_{k=1}^{\infty}$ of nonnegative numbers such that $b_k^{(q)} \leq a_k$ for all k, $\sum b_k^{(q)} \leq 1$, and $\sum a_k^{1/q} [b_k^{(q)}]^{1-1/q} = \infty$.

For each positive integer k, we let $e_k = \sum_{q=2}^{\infty} 2^{-q} b_k^{(q)}$, so that $e_k \leq a_k$, and for each positive integer n, $\sum_{k=1}^{n} e_k = \sum_{k=1}^{n}\sum_{q=2}^{\infty} 2^{-q} b_k^{(q)} = \sum_{q=2}^{\infty} 2^{-q}[\sum_{k=1}^{n} b_k^{(q)}] \leq 1$.

If p is in $(0, 1)$, then there is a positive integer q such that $q > 1$ and $1/q < p$, so that $\sum a_k^p e_k^{1-p} \geq \sum a_k^{1/q} e_k^{1-1/q} \geq (2^{-q})^{1-1/q} \sum a_k^{1/q}(b_k^{(q)})^{1-1/q} = \infty$.

3. **Preliminary definitions and theorems concerning real-valued interval functions.** Throughout this paper all integrals discussed are Hellinger (2) type limits of the appropriate sums.

Suppose $[a, b]$ is a number interval.

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