SUMS OF SERIES INVOLVING BESSEL COEFFICIENTS

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1. **Introduction.** The Bessel coefficient of the first kind is defined by its generating function as

$$e^{-z(r-r^{-1})/2} = \sum_{n=-\infty}^{\infty} r^n J_n(-z),$$

which is valid for all values of z and $r(r \neq 0)$. In connection with some unpublished work of the author, it was shown that Kepler's equation of celestial mechanics

$$M = E - \epsilon \sin E$$

leads to the pair of inverse functions

$$\text{(I)} \qquad s^k = \sum_{p=-\infty}^{\infty} r^p J_{k-p}(k\epsilon)$$

$$\text{(II)} \qquad r^p = \sum_{k=-\infty}^{\infty} \frac{p}{k} J_{k-p}(k\epsilon) s^k,$$

where

$$r = e^{iE}, \qquad s = e^{iM}$$

and

$$\frac{p}{k} J_{k-p}(k\epsilon) = 0 \quad \text{when} \quad k = 0.$$

The first of these, (I), is obtained immediately from the generating function alone, while the derivation of (II), which is equivalent to the Fourier series expression of e^{ipE} in terms of M is, for example, in [1; 71 ff].

Expansion II represents a variation of the Kapteyn series which is defined as

$$\sum_{n=0}^{\infty} a_n J_{m+n}[(m+n)\epsilon],$$

in which m and the coefficients a_n are constants. The index of summation affects the coefficient a as well as the order and argument of the Bessel coefficient of each term of the sum. The same is true in (II); however, the argument does not contain p. In (I) the argument is independent of the index of sum-

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