INVARIANT MEANS AND INVARIANT MATRIX METHODS OF SUMMABILITY

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0. Introduction. Let N be the set of the positive integers (though in Part I a more general set is permitted), E the normed linear space of bounded real-valued functions on N, E' the conjugate space of E. If $\sigma: N \to N$ is a *motion* (§ 2), it induces a linear operator S defined by $(Sf)(p) = f(\sigma p)$. An element $\phi' \in E'$ is called a *σ-mean* if $\| \phi' \| = (\phi', u) = 1$ (where u is the constant function with value 1), and if $(\phi', f) = (\phi', Sf)$ for all $f \in E$. In case σ is the translation mapping $n \to n + 1$, a σ-mean is often called a *Banach limit* [1; 33–34]. Let M'_σ denote the set of all σ-means.

If σ and μ are two motions, conditions under which $M'_\sigma \subset M'_\mu$ were discussed in [4; Theorem 3.3, 4.2]. In Part I below, following several sections of definitions and lemmas required throughout this paper, an improvement of [4; Theorem 4.2] is presented in Theorem 10. It should be noted here that in [4] the hypothesis 'σ and τ have the same sets of finite cycles' was mistakenly omitted from Theorems 3.3 and 4.2. Lemma 4.1 of [4], though incorrect as it stands, is true if the set in question intersects no finite cycles. Only this form of the lemma is used, in the proof of Theorem 4.2 as amended. The rest of [4] is unaffected, as it uses only the amended versions of Theorems 3.3 and 4.2, and will be referred to freely in the present paper.

In Part II the stringency of the condition of Theorem 10 is perhaps dramatized by the exhibition of a non-denumerable collection of motions $\{\sigma_r\}$ chosen so that if $r \neq s$, then $M'_{\sigma_r} \cap M'_{\sigma_s}$ is empty.

Part III points out that all σ-means, for any motion σ, are extensions of the functional 'limit' defined for all convergent sequences, and the digression in §§ 20 and 21 shows that when σ has finite orbits, or even when it is entirely made up of finite cycles, the class of σ-means contains some elements which do not extend 'limit', and some which do.

The remainder of Part III and all of Part IV then form a commentary on, and extension of, part of a paper by G. G. Lorentz [6]. Lorentz calls a function $f \in E$ *almost convergent* if (ϕ', f) has the same value for all Banach limits ϕ'. For such functions, the assignment of (ϕ', f) is a method of summability (though not a matrix method), and Lorentz relates this method to certain classes of matrix methods.

It is shown in Parts III and IV that there corresponds to each motion σ a summability method V_σ analogous to the Lorentz method (which corresponds to translation), that each $f \in E$ is summable by at least one of the methods V_σ;

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