A DEFINITE INTEGRAL

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The aim of the paper is to prove the following formula:

$$(1)\qquad \int_0^\infty e^{-xs}(\sinh x)^{2\lambda} P_m^{(\lambda)}(\cosh x)\,dx$$

$$= \frac{s\lambda\Gamma(\frac{1}{2}(s-m)-\lambda)\Gamma(2\lambda+m)\Gamma(\frac{1}{2}(s+m))}{2^{2\lambda+1}\Gamma(m+1)\Gamma(\frac{1}{2}(s-m)+1)\Gamma(\frac{1}{2}(s+m)+\lambda+1)}$$

where s is a complex number with real part $> m + 2\lambda$, λ is a real number $> -\frac{1}{2}$ and $P_m^{(\lambda)}(u)$ is the ultraspherical polynomial of degree m.

First we prove some lemmas. Let

$$(a)_q = a(a+1)\cdots(a+q-1).$$

Then

$$\frac{\Gamma(a+q)}{\Gamma(a)} = (a)_q\,, \qquad \frac{\Gamma(a)}{\Gamma(a-q)} = (-1)^q(1-a)_q\,,$$

and

$$2^{-2q}\,\frac{\Gamma(a+2q)}{\Gamma(a)} = (\tfrac{1}{2}a)_q(\tfrac{1}{2}a+\tfrac{1}{2})_q\,.$$

The definition of generalized hypergeometric series [1; 8] is

$${}_{p+1}F_p\left[\begin{matrix}\alpha_1\,, & \cdots\,, & \alpha_{p+1}\,; z \\ \beta_1\,, & \cdots\,, & \beta_p\end{matrix}\right] = \sum_{q=0}^\infty \frac{(\alpha_1)_q\cdots(\alpha_{p+1})_q}{q!\,(\beta_1)_q\cdots(\beta_p)_q}\,z^q.$$

This series is absolutely convergent if $|z| < 1$.

Lemma 1. *If $s > l$, then*

$$\int_0^\infty e^{-xs}(\sinh x)^l\,dx = \frac{\Gamma(l+1)}{2^{l+1}}\,\frac{\Gamma(\frac{1}{2}(s-l))}{\Gamma(\frac{1}{2}(s+l)+1)}.$$

Proof. Put $e^{-2x} = y$. The above integral is equal to

$$\frac{1}{2l}\int_0^\infty e^{-(s-l)x}(1-e^{-2x})^l\,dx = \frac{1}{2^{l+1}}\int_0^1 y^{\frac{1}{2}(s-l)-1}(1-y)^l\,dy$$

$$= \frac{1}{2^{l+1}}\,\frac{\Gamma(l+1)\Gamma(\frac{1}{2}(s-l))}{\Gamma(\frac{1}{2}(s+l)+1)}.$$

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