THE INTEGRAL OF A GENERALIZED ALMOST PERIODIC FUNCTION

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We shall denote by B the class of functions almost periodic in the sense of Besicovitch and by I_B the class of integrals of functions of B, i.e. the class of functions $F(x)$ of the form $F(x) = \int_0^x f(t)\,dt$, where $f(x) \in B$. If B_0 and S are the classes of functions almost periodic in the sense of Bohr and Stepanoff, we define in an analogous way I_{B_0} and I_S.

We first prove (Theorem 1) the following statement:

Let $F(x) \in I_B$. In order that $F(x) \in B$ it is necessary and sufficient that

$$\lim_{L\to\infty} \bar{M}_x\left\{\left| L^{-1}\int_x^{x+L} F(t)\,dt - L^{-1}\int_0^L F(t)\,dt \right|\right\} = 0.$$

Here $\bar{M}$ denotes the upper mean value.

Next let $C_B(I_B)$ be the closure of the class I_B with the Besicovitch distance, i.e. $F(x) \in C_B(I_B)$ if there exists a sequence of functions $F_m(x) \in B$ such that $\lim \bar{M}\{| F(x) - F_n(x) |\} = 0$, where $n \to \infty$. In a similar way we define $C_{B_0}(I_{B_0})$ and $C_S(I_S)$.

We prove the following theorem:

In order that $F(x) \in C_B(I_B)$ it is necessary and sufficient that $F(x)$ be of the form $F(x) = \int_0^x f(t)\,dt + \phi(x)$, where $f(x), \phi(x) \in B$. This may be written *algebraically*

$$C_B(I_B) = I_B + B.$$

This theorem has analogues for the classes $C_{B_0}(I_{B_0})$ and $C_S(I_S)$. For example see [1]:

$$C_{B_0}(I_{B_0}) = I_{B_0} + B_0\,; \qquad C_S(I_S) = I_S + S.$$

Theorem 1. *Let $f(x) \in B$ and $F(x) = \int_0^x f(t)\,dt$. In order that $F(x) \in B$ it is necessary and sufficient that*

$$\lim_{L\to\infty} \bar{M}_x\left\{\left| L^{-1}\int_x^{x+L} F(t)\,dt - L^{-1}\int_0^L F(t)\,dt \right|\right\} = 0. \tag{1}$$

Proof. The *necessity* of (1) is a general feature of any function $F(x) \in B$. For an easy proof see [2].

Sufficiency. We first show that $F(x)$ is *uniformly B-summable*, i.e. that to every $\epsilon > 0$ we can associate a $\delta > 0$ such that

$$\bar{\mu}(E) < \delta \quad \text{implies} \quad \bar{M}^E\{| F(x) |\} < \epsilon.^{(1)}$$

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