SOME ALMOST POLYHEDRAL WILD ARCS

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An arc in E^3 which is locally polyhedral except at a finite number of points is said to be *almost polyhedral*. The points at which such an arc is not locally polyhedral will be called *singular* points of the arc. It is shown in [5] that there exist uncountably many differently embedded almost polyhedral arcs, each having just one singular point. In these examples, it is important that the singularpoint be an interior point of the arc.

An almost polyhedral arc whose only singular point is one of its endpoints will be called *nearly* polyhedral. Example 1.2 of [4] is a nearly polyhedral wild arc; we know of no other such example in the literature. The purpose of this paper is to modify the construction of this Fox–Artin arc so as to obtain infinitely many differently embedded nearly polyhedral arcs. We conjecture that there exist uncountably many such arcs, but have not succeeded in proving this. An added feature of our examples is that their wildness is proved by purely geometric arguments.

We define the *penetration index*, $P(A, x)$, of an arc A at a point x of A to be the smallest cardinal number n such that there are arbitrarily small 2-spheres enclosing x and containing no more than n points of A. This is an extension of O. G. Harrold's notion of local peripheral unknottedness (Property $\mathcal{P}$ of [2]). For a nearly polyhedral arc, the only penetration index which is of interest is that at its singular end point. Consequently, if A is locally polyhedral except at q, we will use $P(A)$ for $P(A, q)$.

The Fox–Artin arc mentioned above will be designated A_1 ; our examples consist of arcs A_2 , A_3 , $\cdots$ such that for each n, $P(A_n) = 2n + 1$. The arguments also show that $P(A_1) = 3$ and hence give an alternative proof of the wildness of A_1 .

Since the construction of our examples so closely parallels that of A_1 given in [4], most of the details of the construction will be omitted. We start with a solid right circular cylinder C, points a_1 , a_2 , $\cdots$ a_{2n+1} on one base and points b_1 , b_2 , $\cdots$ b_{2n+1} on the other. These points are joined by disjoint polygonal oriented arcs K_1 , K_2 , $\cdots$ K_{2n+1} as shown in Fig. 1; let $K = \bigcup_{i=1}^{2n+1} K_i$. For convenience in the proof, we construct these arcs so that there is a disk G lying in a horizontal plane and bounded by K_{n+1} and the (straight) interval $a_{n+1}a_{n+2}$, such that $G \cap (K - K_{n+1})$ is a single point x and such that there is a vertical interval e lying in K_{2n+1} and containing x as an interior point.

The arc A_n is obtained by fitting together an infinite number of copies of K, together with $n + 1$ additional arcs, in a manner entirely analogous to that

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