SOME THEOREMS CONCERNING THE DUAL OF A COMMUTATIVE SEMIGROUP

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1. **Introduction.** An n-dimensional representation of a semigroup S is a homomorphism of S into the multiplicative semigroup of $n \times n$ complex matrices, $n \geq 1$. The dual of a commutative semigroup S is a set of 1-dimensional representations and is an analogue of the dual of an abelian group. A number of interesting properties have been proved for representations of semigroups. For example, the dual of a commutative semigroup S may be identified with the set of all non-trivial multiplicative linear functionals on the algebra $l_1(S)$, which in turn is in $(1 - 1)$ correspondence with the space of all regular maximal ideals of $l_1(S)$ (see [2, 2.7]). Moreover, the condition that the dual of S is a separating family on S is equivalent to semisimplicity of the algebra $l_1(S)$ [2, 3.5]. Various other properties of the dual of S have been established, and in some cases its structure has been described. Some of these properties have their analogues in the theory of characters of abelian groups.

The purpose of this paper is to establish several properties of the dual of a commutative semigroup S. In particular, we give necessary and sufficient conditions for a complex-valued function χ to be a semicharacter of S in terms of faces of S and the dual of commutative semigroups which satisfy the cancellation law (and may be groups). Furthermore, we establish a corresponding result for semicharacters χ for which $| \chi(x) | = 1$ or 0 for all $x \in S$. We use the Hewitt-Zuckerman decomposition of a commutative semigroup to ramify the above results. We also consider several special cases.

By a semigroup we mean a non-empty set on which an associative multiplication is defined. Throughout the whole paper S will denote an arbitrary commutative semigroup. A bounded complex-valued function χ on S satisfying the functional equation $\chi(x)\chi(y) = \chi(xy)$ for all $x, y \in S$ is called a semicharacter of S if it is not identically zero [1, 1.3]. It is clear that if S is a group, then χ is a character of S. A non-empty subset F of S is called a face of S if, for all $u, v \in S$, $uv \in F$ if and only if $u, v \in F$ [1, 3.4]. Clearly F is a semigroup. We will use the following notations: $S^\wedge$ will denote the set of all semicharacters of S, the dual of S; $A(S)$ the set of all semicharacters χ of S such that $| \chi(x) | = 1$ or $\chi(x) = 0$ for all $x \in S$; $B(S)$ all semicharacters of S vanishing nowhere; $C(S)$ all semicharacters of S such that $| \chi(x) | = 1$ for all $x \in S$. If F is a face of S, then $D(F)$ will denote all semicharacters of S vanishing exactly off F, and $E(F)$ all semicharacters χ of S such that $| \chi(x) | = 1$ if $x \in F$ and $\chi(x) = 0$ if $x \notin F$.

Under certain conditions $S^\wedge$ is a semigroup under multiplication $\chi_1\chi_2(x) = \chi_1(x)\chi_2(x)$ for all $\chi_1, \chi_2 \in S^\wedge$ and all $x \in S$. Whenever $S^\wedge$ or a subset of it is

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