COMPOSITION RINGS

By Irving Adler

1. Definitions and summary.

1.1. Let $R = A[x]$ be the ring of polynomials over a ring A. If $p, q \in R$, then $p \circ q \in R$, where $(p \circ q)(x) = p(q(x))$. The composition operation, denoted by $\circ$, has these properties:

$$(C1) \qquad (f + g) \circ h = f \circ h + g \circ h$$

$$(C2) \qquad (fg) \circ h = (f \circ h)(g \circ h) \qquad (f, g, h \in R)$$

$$(C3) \qquad f \circ (g \circ h) = (f \circ g) \circ h$$

We use this fact as the point of departure for defining an abstract algebraic structure with three binary operations:

Definition. R is a *composition ring* if it is a commutative ring, not necessarily with 1, and a binary operation $\circ$ is defined in R satisfying axioms C1, C2, and C3.

If R contains an identity for the operation $\circ$, we shall denote it by I.

Definition. c is a *constant* if $c \circ f = c$ for all $f \in R$. If N is any subset of R, the set of all constants in N is called the *foundation* of N, and is denoted by Found N.

A composition ring is essentially the same as the "tri-operational algebra" treated by Menger, Mannos, et al. However, they used a different notation and slightly different axioms. Menger assumed for his tri-operational algebra that it contains an identity I, that $I \neq 1$, and that 1 is a constant. He also tacitly assumed that the algebra was an integral domain. Mannos dispensed with these restrictive assumptions, and also eliminated the assumption that the ring was commutative and with unity element, to obtain what he called a T-0 algebra. A composition ring with identity and in which 1 is a constant he called a T^*-0 algebra.

1.2. *Examples of composition rings.*

1. R is any commutative ring, and $\circ$ is defined by $r \circ s = 0$ for all $r, s \in R$. In this case we shall call R a *null* composition ring.

2. R is any commutative ring, and $\circ$ is defined by $r \circ s = r$ for all $r, s \in R$. Then the foundation of R is R. In this case we shall call R a *constant* composition ring. A composition ring is called *trivial* if it is constant or null.

3. Let K be a commutative ring. Let $R = K^K$ (the ring of all functions

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