ORDINARY LINEAR DIFFERENTIAL OPERATORS OF MINIMUM NORM

By William T. Reid

1. Introduction. The present paper is concerned with linear vector ordinary differential operators of the form

$$(1.1) \qquad L[y] \equiv A_1(t)y'(t) + A_0(t)y(t) + b(t), \qquad t \,\varepsilon\, I: \; a \leq t \leq b,$$

where $y(t) \equiv (y_i(t))$, $(i = 1, \cdots, n)$, belongs to the class $\mathfrak{A}(I)$ of a.c. (absolutely continuous) vector functions on I, while the $n \times n$ coefficient matrices $A_0(t)$, $A_1(t)$, and n-dimensional vector function $b(t)$ are such that $A_1(t)$ is non-singular, with $A_1^{-1}(t)A_0(t)$ and $A_1^{-1}(t)b(t)$ (Lebesgue) integrable on this interval.

For $B(t)$ an $n \times n$ matrix measurable on I, and various Lebesgue linear normed function spaces $\mathfrak{L}$ of n-dimensional vector functions, there are characterized the $x(t) \,\varepsilon\, \mathfrak{L}$ of minimum norm $\mathfrak{M}[x]$, and belonging to the set Γ defined by

$$(1.2) \qquad \Gamma = \{x(t) \mid \exists \; y(t) \,\varepsilon\, \mathfrak{A}(I) \;\; with \;\; L[y] = B(t)x(t), \, y(a) = \xi^a, \, y(b) = \xi^b\},$$

where ξ^a and ξ^b are given n-dimensional constant vectors. In particular, if $B(t)$ is non-singular on I, the problem is that of determining $y(t) \,\varepsilon\, \mathfrak{A}(I)$ satisfying $y(a) = \xi^a$, $y(b) = \xi^b$ and such that $B^{-1}(t)L[y]$ is an element of $\mathfrak{L}$ of minimum norm. Another important instance is that of problems involving matrices $B(t) \equiv \| B_{ij}(t) \|$, $(i, j = 1, \cdots, n)$, such that

$$(1.3) \qquad B_{ij}(t) \equiv 0, \qquad (i \neq j), \qquad B_{ii}(t) = \delta(t, J^i), \qquad (i = 1, \cdots, n),$$

where $\delta(t, J^i)$ is the characteristic function of a measurable subset J^i of I' and at least one of the J^i is of positive measure. In this case $B(t)$ is idempotent on I, $(B(t) \equiv B^2(t))$, and norm-reducing, $(\mathfrak{M}[Bx] \leq \mathfrak{M}[x])$, on each of the Lebesgue spaces considered, and consequently the problem of determining $x(t) \,\varepsilon\, \mathfrak{L}$ of minimum norm is solved by finding a $y(t)$ of

$$\{y(t) \mid y(t) \,\varepsilon\, \mathfrak{A}(I), \, L_i[y] = 0 \;\; on \;\; I - J^i, \, (i = 1, \cdots, n)\},$$

such that $L[y] = (L_i[y])$ is an element of $\mathfrak{L}$ of minimum norm. For a particular Lebesgue function space, and real-valued operators (1.1), this latter problem has been treated by Carter [3].

In §2 it is shown that the above described problem is equivalent to the determination of an "extremal" solution of a corresponding finite moment problem, to which the general results of Hahn and Banach on linear functionals, (see, for example, Dunford and Schwartz [5; 86]), are applicable. For the Lebesgue function spaces under consideration the explicit solution of this

Received August 28, 1961. This research was sponsored by the Mathematical Sciences Directorate, Air Force Office of Scientific Research of the Air Research and Development Command, under Contract AF 49(638)–994.