TWO PROBLEMS OF HEWITT

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In this paper the solutions to two problems raised by Edwin Hewitt [1; 332, 11.28–32] are presented. We shall follow the terminology and notations used by Hewitt.

THEOREM 1. *If τ is an infinite cardinal number, there exists a T_1 space R with $\Delta(R) = \tau$ such that no expansion S of R is a Hausdorff space with $\Delta(S) = \Delta(R)$.*

Proof. Let E be a set with cardinality τ and $T(E)$ be the T_1 space defined on E whose open sets are the empty set and subsets of E with finite complements. Clearly $\Delta(T(E)) = \tau$. Hewitt [1, Theorem 13] has proved that there exists a τ-maximal expansion S^* of $T(E)$ which is necessarily a T_1 space. Let a be any point in S^* and $\mathfrak{U}_a = \{U(a)\}$ be the set of all open neighborhoods of a in S^*.

Now let α be any element not in E and let $\mathfrak{U} = \mathfrak{O}(S^*) \cup \{U(a) - \{a\} \cup \{\alpha\} \mid U(a) \in \mathfrak{U}_a\}$ be a set of subsets of $E^* = E \cup \{\alpha\}$. If the sets $U \in \mathfrak{U}$ are now defined to be open neighborhoods of each of the points they contain, it is clear that $\mathfrak{U}$ satisfies the first three neighborhood axioms of Hausdorff. Hence, $\mathfrak{U}$ defines a topological space R. It is easy to show that R is T_1 and that $\Delta(R) = \tau$. Since every neighborhood of α intersects every neighborhood of a, R is not a Hausdorff space.

Suppose there is a Hausdorff expansion S of R with $\Delta(S) = \Delta(R)$. Then there exist in S disjoint open neighborhoods G, H of a and α respectively. Let E, considered as a subspace of R (or S) be designated by E_R (or E_S). Then since $E \in \mathfrak{O}(S)$ and $G \subset E$, $G \in \mathfrak{O}(E_S)$. From the above discussion it follows that E_S is an expansion of E_R. But E_R is the space S^*. Thus E_S is an expansion of S^*. Furthermore $\Delta(E_S) = \tau$ since $\Delta(S) = \tau$ and since the removal of one point does not affect the cardinality of an infinite set. But since S^* is τ-maximal, this means that E_S is no proper expansion of S^*. That is, that $\mathfrak{O}(E_S) = \mathfrak{O}(S^*)$. In particular then, since $G \in \mathfrak{O}(E_S)$, $G \in \mathfrak{O}(S^*)$. Hence $G \in \mathfrak{U}_a$. Since every non-empty open set of S has cardinality τ, the cardinality of $H \cap U(\alpha)$ is τ. But $U(\alpha) = U(a) - \{a\} \cup \{\alpha\}$ and since $G \in \mathfrak{U}_a$ there is some $U_1(\alpha) = G - \{a\} \cup \{\alpha\}$. Thus the cardinality of $H \cap (G - \{a\} \cup \{\alpha\})$ is τ. Hence the cardinality of $H \cap G$ is τ since the addition or removal of a finite number of points does not affect the cardinality of an infinite set. This contradiction completes the proof.

THEOREM 2. *If τ is an infinite cardinal number and R is a regular space with $\Delta(R) = \tau$, there exists a completely regular space S which is an expansion of R with $\Delta(S) = \Delta(R)$.*

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