THE MAXIMUM NUMBER OF ZEROS IN THE POWERS OF AN INDECOMPOSABLE MATRIX

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I. Introduction. Let A be an n-square matrix with complex entries. A is called *decomposable* if there is a permutation matrix P such that PAP^T is a subdirect sum. Otherwise A is called *indecomposable.*

In a recent conversation Dr. Seymour Haber posed the following question. Given an n-square indecomposable matrix A with complex entries, how many fixed positions (i, j), $1 \leq i, j \leq n$, can be zero in every positive integral power of A? This problem has significance in certain combinatorial problems and, as will subsequently be shown in case A is normal, reduces to a familiar kind of question: namely, given an integral matrix H what kinds of 0, 1 matrices B (if any) exist satisfying $BB^T = H$?

We remark that in order to check whether $(A^k)_{ij} = 0$, $i \neq j$, $k = 1, 2, \cdots$, it suffices to examine $A, A^2, \cdots, A^{n-1}$ (Cayley-Hamilton).

As an example, let A be indecomposable, with non-negative entries, and positive trace. If $m \geq 2n - 2$, then each entry of A^m is positive [1]. On the other hand, if $P_n = (p_{ij})$ denotes the n-square full cycle permutation matrix defined by

$$p_{i1} = \delta_{in}, \qquad p_{ij} = \delta_{i+1,j} \quad \text{if} \quad j > 1,$$

then the (i, j) entry of $(P_n)^k$ is both zero and one for infinitely many values of k. Of course, P_n is indecomposable.

In general, the question seems difficult to answer. However, in case A is an indecomposable normal matrix with distinct eigenvalues, our main result yields a realistic upper bound for the number of fixed positions that can be zero in every positive integral power of A (Theorem 4).

II. The combinatorial problem. Let $x_1, \cdots, x_t$ be n-vectors, and denote by $\langle x_1, \cdots, x_t\rangle$ the space spanned by $x_1, \cdots, x_t$. If $Ax_i \in \langle x_1, \cdots, x_t\rangle$, $i = 1, \cdots, t$, then $\langle x_1, \cdots, x_t\rangle$ is called an *invariant subspace* under A. We put $\epsilon_\alpha = (\delta_{\alpha 1}, \delta_{\alpha 2}, \cdots, \delta_{\alpha n})$, $\alpha = 1, \cdots, n$.

We have immediately from the definition the

Lemma. *A is decomposable if and only if for some k, $1 \leq k \leq n$, $\langle \epsilon_{i_1}, \cdots, \epsilon_{i_k}\rangle$ is an invariant subspace under A.*

If A is normal, then A^*, the conjugate transpose of A, is a polynomial in A. Denote by $Z(A)$ the set of positions (i, j), $1 \leq i, j \leq n$, for which $(A^k)_{ij} = 0$, $k = 1, 2, \cdots$. When A is normal, we see that for $i \neq j$, $(i, j) \in Z(A)$ if and

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