TWO-POINT BOUNDARY PROBLEMS FOR LINEAR SELF-ADJOINT DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER WITH MIDDLE TERM

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The fourth-order differential equation under consideration is

$$[(r(x)y'')' + q(x)y']' - p(x)y = 0 \quad \text{on} \quad [a, \infty), \tag{1}$$

where $r(x)$ is positive and all three coefficients are continuous. Most of the recent investigations [1], [2], [6], [7] have required that $q(x) \equiv 0$ and that $p(x)$ have a constant sign, usually positive. Also, Leighton and Nehari [7] pointed out the striking differences in distribution of zeros when $p > 0$ and when $p < 0$. They also gave a new transformation (see §1) for removing the middle term of (1) when the second-order equation

$$(r(x)y')' + q(x)y = 0$$

is disconjugate. Many cases remain where the middle term cannot be conveniently eliminated and where the needed information about distribution of zeros can be obtained directly from the full equation (1). The author began such a study in [3] for the case where $q \geq 0$ and $p \geq 0$ by means of a systems-formulation of Sternberg [8], [9]. In the present paper further results are obtained with some relaxation of the nonnegativeness of p and q, particularly of q.

This study will be concerned with the existence of solutions of (1) which satisfy the double-zero initial conditions at $x = a$

$$y(a) = y'(a) = 0 \qquad \text{(clamped end)} \tag{2}$$

and one of the following conditions at a second value $b \in (a, \infty)$:

(3) $\quad y(b) = y'(b) = 0$ (clamped end) [1], [2], [3], [6], [7], [9]

(4) $\quad y_1(b) = y_2(b) = 0$ (free end) [1], [3], [6]

(5) $\quad y'(b) = y_1(b) = 0$ or $y(b) = y_2(b) = 0$

(6) $\quad y'(b) = y_2(b) = 0$

(7) $\quad y(b) = y_1(b) = 0$ (supported ends) [2]

where, throughout the paper, for a function $y(x)$ having suitable derivatives:

$$y_1 = r(x)y'' \quad \text{and} \quad y_2 = y_1' + q(x)y'. \tag{8}$$

In this notation, equation (1) becomes $y_2' = py$.

Received September 16, 1961. Sponsored by the United States Army under Contract No. DA-11-022-ORD-2059, Mathematics Research Center, United States Army.