A GENERALIZED LEBESGUE COVERING THEOREM

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1. **Introduction and results.** We start with the Lebesgue Covering Theorem [1; 42, Theorem IV 2]:

Suppose an n-dimensional cube is the union of a finite number of closed sets, none of which contains points of two opposite faces. Then at least $n + 1$ of these closed sets have a common point.

If the word "finite" in the hypothesis is omitted, the resulting statement is false. However, if "finite" is replaced by "countable," the resulting statement is plausible, and in fact is true for $n = 1$. The author is indebted to Dr. Hans Debrunner for calling his attention to this problem. The present paper gives counter examples (in which the sets of the covering are also convex) to show that the statement is false for $n > 1$. But with the added hypothesis that the sets of the covering form a *contracting sequence* the statement becomes true. (In a compact metric space a collection of point sets is *contracting* if and only if for every $\alpha > 0$ only a finite number of the sets have diameter greater than α. This is a special case of the notion *topologically contracting*, due to R. L. Moore [2; 341].) In fact we have the following theorem, which is the principal result of this paper:

Theorem 1.1. *Suppose X is a compact metric space and* $\dim X \geq n$. *If d is a metric for X, then there exists an $\epsilon > 0$ such that if $\mathcal{G}$ is a contracting sequence of closed sets covering X and $d(G) < \epsilon$ for all $G \in \mathcal{G}$, then there exists a point p such that p is common to at least $n + 1$ elements of $\mathcal{G}$. If X is the unit n-cell and d the Euclidean metric, then ϵ may be taken equal to* 1.

2. **Examples.** In this section we prove the following two theorems.

Theorem 2.1. *If X is a compact metric space (metric d),* $\dim X \leq n$ *(n a positive integer), then for every $\epsilon > 0$ there exists a countable closed cover $\mathcal{G}$ of X such that $d(G) < \epsilon$ for all $G \in \mathcal{G}$ and no three elements of $\mathcal{G}$ have a point in common. Thus if $n \geq 2$, no $n + 1$ elements of $\mathcal{G}$ have a point in common.*

Theorem 2.2. *As in Theorem* 2.1, *if X is the n-cell I^n, then the elements of $\mathcal{G}$ may all be taken to be convex closed sets.*

Construction 2.2. *Step* 1. Using n finite families of hyperplanes parallel to the various sides of I^n we construct an n-dimensional cellular complex K, such that (1) $\cup K = I^n$, (2) if $t \in K$, then $d(t) < \epsilon$, (3) if $s \in K$, $t \in K$, and $s \neq t$, then

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