GENERATING FUNCTIONS FOR POWERS OF CERTAIN SEQUENCES OF NUMBERS

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1. **Introduction.** The Fibonacci numbers f_n may be defined by

$$f_0 = f_1 = 1, \qquad f_n = f_{n-1} + f_{n-2} \qquad (n \geq 2).$$

Their generating function is

$$f_1(x) = \sum_{n=0}^{\infty} f_n x^n = (1 - x - x^2)^{-1}.$$

More generally we may put

$$f_k(x) = \sum_{n=0}^{\infty} f_n^k x^n.$$

Riordan [2] has proved that $f_k(x)$ satisfies the following recurrence relation:

$$(1.1) \qquad (1 - a_k x + (-1)^k x^2) f_k(x) = 1 + kx \sum_{j=1}^{[k/2]} (-1)^j \frac{a_{kj}}{j} f_{k-2j}((-1)^j x) (k \geq 1),$$

where

$$a_1 = 1, \qquad a_2 = 3, \qquad a_k = a_{k-1} + a_{k-2} \qquad (k \geq 3)$$

and a_{kj} is defined by means of

$$(1 - x - x^2)^{-j} = \sum_{k=2j}^{\infty} a_{kj} x^{k-2j}.$$

In the present paper we consider the numbers u_n defined by

$$u_0 = 1, \qquad u_1 = p, \qquad u_n = pu_{n-1} - qu_{n-2} \qquad (n \geq 2),$$

where $p^2 - 4q \neq 0$. We put

$$u_k(x) = \sum_{n=0}^{\infty} u_n^k x^n,$$

so that

$$u_1(x) = (1 - px + qx^2)^{-1}.$$

Generalizing (1.1) we show first that

$$(1.2) \qquad (1 - v_k x + q^k x^2) u_k(x) = 1 + kx \sum_{r=1}^{[k/2]} \frac{1}{r} q^r a_{kr} u_{k-2r}(q^r x) \qquad (k \geq 1),$$

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