# PSEUDO-COMPLEMENTS IN SEMI-LATTICES

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There already exists a well-developed theory of pseudo-complements in lattices, and particularly in distributive lattices, due to V. Glivenko [4a], M. H. Stone [8a], [8b], and Garrett Birkhoff [1]. We show here that a large part of this theory may be extended to pseudo-complemented semi-lattices, without making use of the join operation, or even assuming that it exists. In particular, the result of Glivenko that the closed elements form a Boolean algebra still holds. The use of only one of the lattice operations yields a theory that is both simpler and more general.

G. Grätzer and E. T. Schmidt have independently obtained similar results concerning pseudo-complemented semi-lattices, which have not yet been published. A theory of *relatively* pseudo-complemented semi-lattices has been developed by A. Monteiro [6]. The theory of pseudo-complements has applications to Brouwerian logics, to the theory of ideals in Boolean algebras, and in topology to the lattice of all open sets and the Boolean algebra of all regular open sets.

The pseudo-complement $b^*$ of an element $b$ is the greatest element disjoint from $b$, if such an element exists. The defining property of $b^*$ is:

$$a \cap b = 0 \Leftrightarrow a \cap b^* = a. \tag{4}$$

Since in a lattice or semi-lattice the order relation is defined by $a \leq b \Leftrightarrow a \cap b = a$, (4) is equivalent to:

$$a \cap b = 0 \Leftrightarrow a \leq b^*. \tag{4a}$$

Since (4) involves only the meet operation $a \cap b$, it has meaning in a semi-lattice, in which the join operation is not defined. The notion of pseudo-complement is not self-dual. The dual notion, that of the quasi-complement $b^+$ of an element $b$, is defined by

$$a \cup b = I \Leftrightarrow a \cup b^+ = a. \tag{4d}$$

If every element of a lattice has a pseudo-complement (necessarily unique), then the lattice is said to be a *pseudo-complemented lattice.* A lattice is said to be *relatively pseudo-complemented* if every pair of elements $a$ and $b$ have a relative pseudo-complement $c$ such that $a \cap x \leq b \Leftrightarrow x \leq c$.

Garrett Birkhoff in his book *Lattice Theory* [1] shows that every relatively pseudo-complemented lattice is also distributive. (This book will be referred to hereafter as LT). However, a merely pseudo-complemented lattice need not be distributive or even modular. This is shown by the example of the five-

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