THE SCHOLZ-BRAUER PROBLEM IN ADDITION CHAINS

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1. **Introduction.** Following A. Scholz [3], we say that a sequence of integers

$$1 = a_0 , \quad a_1 , a_2 , \cdots , a_r = n$$

is an addition chain for the positive integer n provided for each $i > 0$ we have

$$(1.1) \qquad a_i = a_j + a_k \quad \text{for some} \quad j, k < i \qquad (j = k \quad \text{is allowed}).$$

The integer r is called the length of the chain. The smallest possible value of r is denoted $l(n)$. Our interest here is in the *shortest* chains for n, i.e. chains of smallest possible length.

Scholz proposed and Alfred Brauer proved that

$$(1.2) \qquad q + 1 \leq l(n) \leq 2q, \quad \text{for} \quad 2^q + 1 \leq n \leq 2^{q+1}, \qquad q \geq 1$$

$$(1.3) \qquad l(ab) \leq l(a) + l(b).$$

Scholz conjectured that $l(2^q - 1) \leq l(q) + q - 1$, $q \geq 1$. We will refer to this in the sequel as *Scholz's conjecture.* This conjecture has not yet been completely solved for general values of q. Partial solutions have been offered by Brauer [1], W. R. Utz [4], and Walter Hansen [2].

Brauer proved that Scholz's conjecture is true provided that among the shortest chains of q, there is at least one satisfying

$$(1.4) \qquad a_i = a_{i-1} + a_j , \quad \text{some} \quad j < i \qquad (i = 1, 2, \cdots , r).$$

(We refer to any chain satisfying (1.4) as a *special chain of type* A.) The minimal length of a special chain of type A for n is denoted $l^*(n)$. Hansen has shown that there are integers n for which $l^*(n) > l(n)$; thus, Scholz's conjecture is *not* proved by arguing that among the chains of shortest length there is one of type A.

Utz has shown that Scholz's conjecture is true whenever q has the form $q = 2^t$ or $q = 2^s + 2^t$, $s > t \geq 0$, s and t integral.

In this paper we will extend Utz's results by showing that the conjecture holds when q is of the form $q = 2^{c_1} + 2^{c_2} + 2^{c_3}$, $c_1 > c_2 > c_3 \geq 0$. In attempting to extend the result to the case when q is of the form $2^{c_1} + 2^{c_2} + 2^{c_3} + 2^{c_4}$, we encountered some difficulties which we could not completely resolve. Our results are contained in Theorem 2. The question to be settled here is the value of $l(n)$ when $n = 2^{c_1} + 2^{c_2} + \cdots + 2^{c_i}$, $c_1 > c_2 > \cdots > c_i \geq 0$. When $i = 1$, $l(n) = c_1$; when $i = 2$, $l(n) = c_1 + 1$; when $i = 3$, $l(n) = c_1 + 2$. However, when $i = 4$, $l(n)$ is $c_1 + 2$ is some cases and $c_1 + 3$ in others. We are not able to distinguish these two cases completely.

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