GENERALIZATION OF AN INTEGRAL FORMULA OF BATEMAN

By H. W. Gould

1. Introduction. Among the many interesting relations discovered by the late Harry Bateman in his work with special functions we single out his celebrated integral formula for Bessel functions

$$J_{a+c}(t) = c \int_0^1 J_a[t(1-u)]J_c(tu)\,\frac{du}{u} \tag{1.1}$$

which appears in Bateman's paper of 1905 [1; 120].

Special mention of the formula was made by Murnaghan [6; 91] in his obituary of Professor Bateman.

It is interesting to note that Bateman possessed a generalization of the formula which appears in his unpublished manuscript on binomial coefficients [2; 73, 500]. Bateman defines

$$J_a(t, b) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{a+bk}}{2^{a+bk}k!\,\Gamma(a+bk-k+1)} \tag{1.2}$$

which reduces to the ordinary Bessel function when $b = 2$. Bateman shows that this function satisfies the same type formula and in fact

$$J_{a+c}(t, b) = c \int_0^1 J_a[t(1-u), b]J_c(tu, b)\,\frac{du}{u}\cdot \tag{1.3}$$

To prove this he uses a form of the generalized Vandermonde formula

$$\binom{a+c+bn}{n} = \sum_{k=0}^{n} \binom{a+bk}{k}\binom{c+b(n-k)}{n-k}\frac{c}{c+b(n-k)}. \tag{1.4}$$

2. Generalization. It may be of interest to show that the formula of Bateman is quite general and may be extended to a class of coefficients studied by the present writer [3], [4].

We define coefficients C and G by means of the formulas

$$x^a = \sum_{k=0}^{\infty} C_k(a, b)z^k, \qquad z = x^{-b}f(x), \tag{2.1}$$

and

$$x^a g(x, b) = \sum_{k=0}^{\infty} G_k(a, b)z^k, \tag{2.2}$$

Received July 3, 1961. Research supported under National Science Foundation research grant G-14095.