SYNTHETIC PROOF OF SOME PRIME NUMBER INEQUALITIES

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1. Introduction. The simplicity and elegance of Euclid's proof of the infinitude of primes has compelled practically every author of a text on elementary number theory to reproduce the ancient geometer's argument. We recall that Euclid made the ingenious choice of a number $2 \cdot 3 \cdot 5 \cdots p_n + 1$, where p_i denotes the i-th prime number, and he reasoned that

$$\prod_{i=1}^{n} p_i + 1 \geq p_{n+1} . \tag{1.1}$$

Dickson [1] stated that Euler modified Euclid's argument when he considered

$$\varphi\left(\prod_{i=1}^{n} p_i\right) = \prod_{i=1}^{n} (p_i - 1)$$

and concluded that the integers less than $\prod_{i=1}^{n} p_i$ include either primes greater than p_n or have prime factors greater than p_n. Thus Euler used his own function φ to show that there is no greatest prime. Other arguments that may be employed to prove that there are infinitely many primes may be found in [1] and [3].

Uspensky [5] has remarked that the improvement of the inequality (1.1) was very difficult and that the first notable success in this direction was achieved by Tchebycheff who in 1850 proved that there is always a prime between a and $2a - 2$ for any $a > 7/2$. The result above enables one to infer $2p_n > p_{n+1}$ for $n \geq 1$. Tchebycheff's proof, however, was not based solely on the properties of the integers and rational numbers. His and other later proofs relied heavily on properties of irrational numbers, logarithms and the usual concepts of analysis. And, as is well known, practically all information concerning prime number inequalities has been obtained by way of the methods used in analysis and function theory. Proofs in number theory that do not 'involve the usual concepts of analysis—limits, continuity, etc.' are called synthetic proofs [4].

In [5] Uspensky has included a proof of Bonse's inequality: 'a very simple proof of the inequality $p_{n+1}^2 < p_1 \cdot p_2 \cdots p_n$ for $n \geq 4$, due to Bonse. This inequality is not nearly so sharp as $p_{n+1} < 2p_n$, but it maintains a certain interest on account of the simplicity of proof.' The purpose of this paper is to furnish a synthetic proof of a new theorem. Use is made of the theorem to derive some inequalities that are sharper than Bonse's inequality.

2. Main result.

Theorem 1: *If*

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