SYMMETRY IN SPACES OF ENTIRE FUNCTIONS

By Louis de Branges

We are again concerned with Hilbert spaces, whose elements are entire functions, and which have these three properties:

(H1) Whenever $F(z)$ is in the space and has a non-real zero w, the function $F(z)(z - \bar{w})/(z - w)$ is in the space and has the same norm as $F(z)$.

(H2) For every non-real number w, the linear functional defined on the space by $F(z) \rightarrow F(w)$ is continuous.

(H3) Whenever $F(z)$ is in the space, the function $F^*(z) = \bar{F}(\bar{z})$ is in the space and has the same norm as $F(z)$. As usual, if $E(z)$ is an entire function which satisfies

$$|E(\bar{z})| < |E(z)| \tag{1}$$

for $y > 0$, we write $E(z) = A(z) - iB(z)$ where $A(z)$ and $B(z)$ are entire functions which are real for real z and

$$K(w, z) = [B(z)\bar{A}(w) - A(z)\bar{B}(w)]/[\pi(z - \bar{w})].$$

Let $\mathcal{H}(E)$ be the set of entire functions $F(z)$ such that

$$\|F\|^2 = \int |F(t)|^2 |E(t)|^{-2} dt < \infty$$

and

$$|F(z)|^2 \leq \|F\|^2 K(z, z)$$

for all complex z. Then, $\mathcal{H}(E)$ is a Hilbert space of entire functions which satisfies H1, H2, and H3. For each complex number w, $K(w, z)$ belongs to $\mathcal{H}(E)$ as a function of z and

$$F(w) = \langle F(t), K(w, t)\rangle$$

for every $F(z)$ in $\mathcal{H}(E)$. As shown in [7], every Hilbert space of entire functions which satisfies H1, H2, and H3 and which contains a non-zero element is equal isometrically to some such $\mathcal{H}(E)$.

An example of such a space is obtained from the function $E(z) = \exp(-iz)$, in which case the space consists of the entire functions $F(z)$ such that

$$\|F\|^2 = \int |F(t)|^2 dt < \infty$$

and

$$|F(z)|^2 \leq \|F\|^2 \sin(2y)/(2\pi y)$$

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