MONOTONE (NONLINEAR) OPERATORS IN HILBERT SPACE

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1. **Introduction.** Let $\mathfrak{X}$ be a Hilbert space, with real or complex scalars, and with inner product $\langle x, y \rangle$. For $\mathfrak{D} \subset \mathfrak{X}$, we call a (not necessarily linear) operator $F: \mathfrak{D} \to \mathfrak{X}$ a *monotone operator* provided that, for any $x_1, x_2 \in \mathfrak{D}$,

$$\operatorname{Re} \langle x_1 - x_2, F(x_1) - F(x_2) \rangle \geq 0.$$

We prove several properties of such operators, the most important of which are that the "integral equation of the second kind" $x + F(x) = u$, under a few additional hypotheses, always has a solution for x, and that the solution depends continuously on u.

It is the author's feeling that some problems of mathematical physics which involve monotone operators could be better reformulated in terms of the "graphs" of the operators—i.e., in terms of a subset of the product space $\mathfrak{X} \times \mathfrak{X}$. An exploratory reformulation of a "classical" problem was made in [5]. This paper will not, however, be concerned with applications.

2. **Background.** In E^n, let S be a sphere (boundary + interior); we let $r(S)$ denote its radius (which may be zero), and let $\delta(S_1, S_2)$ denote the *distance between the centers* of S_1 and S_2. A theorem of Kirszbraun [3], later rediscovered by Valentine [8], [9] is as follows:

KIRSZBRAUN'S THEOREM. *In E^n, suppose spheres $S_1, \cdots, S_m$ and $S_1', \cdots, S_m'$ are such that (1°) $\bigcap_i S_i \neq \emptyset$, (2°) for $i = 1, \cdots, m$, $r(S_i) = r(S_i')$, and (3°) for $i, j = 1, \cdots, m$, $\delta(S_i, S_j) \geq \delta(S_i', S_j')$. Then $\bigcap_i S_i' \neq \emptyset$.*

An elementary proof has been outlined by Schoenberg [7], and Mickle[4] has given a brief treatment of the extension of this theorem to the case of infinitely many spheres in infinite-dimensional Hilbert space. Since this extension is essential for our development of monotone functions, we give proofs in greater detail.

THEOREM 1. *Let $\mathfrak{X}$ be a Hilbert space (with real or complex scalars), and let A be an index-set. Let S_α and S_α' be spheres, indexed by A, satisfying (1°) $\bigcap_\alpha S_\alpha \neq \emptyset$, (2°) for all $\alpha \in A$, $r(S_\alpha) = r(S_\alpha')$, and (3°) for all $\alpha, \beta \in A$, $\delta(S_\alpha, S_\beta) \geq \delta(S_\alpha', S_\beta')$. Then $\bigcap_\alpha S_\alpha' \neq \emptyset$.*

Proof. Assume (1°), (2°), and (3°), and distinguish an element $\gamma \in A$. It is sufficient to prove that $\bigcap_\alpha (S_\gamma' \cap S_\alpha')$ is nonempty. But the $(S_\gamma' \cap S_\alpha')$ are weakly-closed subsets of the weakly-compact ball S_γ', so by the "finite intersection property", it suffices to show that the intersection of each finite sub-

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