THE WEIERSTRASS TRANSFORM AND HERMITE POLYNOMIALS

By G. G. Bilodeau

1. **Introduction.** The Weierstrass transform $f(x)$ of $\phi(y)$ is defined by

$$f(x) = \int_{-\infty}^{+\infty} k(x - y, 1)\phi(y)\, dy \tag{1.1}$$

where $k(x, t) = (4\pi t)^{-1/2}e^{-x^2/4t}$. In general we write $f(x) = W[\phi]$. Similarly we can define the Weierstrass—Stieltjes transform $f(x) = WS[\alpha]$ as

$$f(x) = \int_{-\infty}^{+\infty} k(x - y, 1)\, d\alpha(y) \tag{1.2}$$

where $\alpha(y)$ is assumed of BV in every finite interval. The basic properties of these transforms have been studied in detail in [5, Chapter VIII]. In particular it was observed that the operator e^{-D^2}, $D \equiv d/dx$, suitably interpreted, will invert $f(x) = W[\phi]$. It is the purpose of this paper to study interpretations of this operator which make strong use of the properties of the Hermite polynomials defined by

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2}.$$

In §2, we add some new results to a familiar interpretation of e^{-D^2} in terms of series by considering the use of special summability methods. In §3, we introduce another interpretation which is intimately connected with the problem of the expansion of a function in Hermite series. In this way we are eventually led to a partial proof of a conjecture of Kogbetliantz [6; 169] on a Borel summability of Hermite Series.

2. **The first interpretation.** Define

$$e^{-D^2}f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(2k)}(x) \tag{2.1}$$

whenever this series converges. The application of this operator to Weierstrass transforms was first mentioned by Eddington [2] and later studied by Hille [4] and Rooney [7]. We quote the following theorem, found in [1; 12], which is a slight improvement over Rooney's result.

Theorem 2.1. *Let $\phi(x)$ be of* BV *in some neighborhood of x_0 and let*

$$\left(\int_{-\infty}^{-a} + \int_{a}^{\infty}\right) y^{-\frac{2}{3}} k(x_0 - y, 2) \mid \phi(y) \mid dy \tag{2.2}$$

Received April 14, 1961. This paper represents a part of the author's thesis written under the direction of Professor D. V. Widder at Harvard University 1959.