# CONVERGENCE OF CONVOLUTION ITERATES OF MEASURES

By H. S. Collins

Recent papers (e.g., see [6]) have given necessary and sufficient conditions for the convergence of the powers of a fixed measure $\nu$ on a compact group. To state the problem more precisely, let $S$ be a compact Hausdorff topological semigroup, and denote by $S^{\sim}$ the convolution semigroup (given the weak-* topology) of non-negative normalized regular Borel measures on $S$. If $\nu \in S^{\sim}$ and $n$ is a positive integer, the notation $\nu^n$ is used for the convolution of $\nu$ with itself $n$ times. The problem to be considered is: can one give simple and useful conditions on $S$ which are equivalent to the convergence of the sequence $\{\nu^n\}$? Several such conditions have been given when $S$ is a group (see the reference listed above), and it is our purpose here to state some new conditions (in Theorem 1). The inclusion of these new conditions allows us to incorporate and give easy proofs as well of all the previously known conditions. Preliminary to our main theorem, however, we state some lemmas of independent interest for compact semigroups $S$. It is emphasized that only the simplest basic theory of compact semigroups is needed for our attack on the problem considered (in direct contrast with previous work, which utilized character, representation, and transform theory).

Let now $S$ be a compact semigroup (this will be the case until just prior to Lemma 5, at which time and subsequently, $S$ is required to be a group). If $\nu \in S^{\sim}$, $\Gamma(\nu)$ denotes the closure of the set of powers of $\nu$ and $K(\nu)$ is the set of cluster points of $\{\nu^n\}$. It is known [4, Theorem 1] that $\Gamma(\nu)$ is a compact (Abelian) semigroup with *kernel* (= minimal ideal) $K(\nu)$, and $K(\nu)$ is a group. It is thus clear that $\{\nu^n\}$ converges if and only if $K(\nu)$ reduces to $\{\lambda\}$, where $\lambda$ is the identity of $K(\nu)$. Further, $K(\nu)$ is the closure of the set of powers of $\nu\lambda = \lambda\nu$ [4, Theorem 4].

We denote by carrier $\nu$ the complement of the largest open set of $S$ having $\nu$ measure zero, and by carrier A (where $A \subset S^{\sim}$) the closure of the set $\bigcup$ {carrier $\mu : \mu \in A$}. The introduction of this notation allows one to talk about the *convergence* of *the sequence* {carrier $\nu^n$}; this by definition means that lim sup carrier $\nu^n$ = lim inf carrier $\nu^n$, where lim sup carrier $\nu^n$ (respectively lim inf carrier $\nu^n$) is the set of points of $S$ each neighborhood of which meets carrier $\nu^n$ for infinitely many $n$ (respectively from some $n$ on). It is easy to see, since carrier $\nu^n$ = (carrier $\nu)^n$ and carrier $\nu^n \cdot$ carrier $\nu^m$ = carrier $\nu^{n+m}$ [2, Lemma 2.1], that lim sup carrier $\nu^n$ is a closed semigroup of $S$. The same statement can be made regarding lim inf carrier $\nu^n$ provided it is non-void, and the containment lim inf carrier $\nu^n \subset$ lim sup carrier $\nu^n$ is obvious.

The following lemma is a generalization to the non-Abelian case of a portion

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