CERTAIN CONVEXITY CONDITIONS ON MATRICES WITH APPLICATIONS TO GAUSSIAN PROCESSES

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We present two theorems concerning matrices which are positive-definite and whose rows satisfy certain convexity conditions (§1). The proofs involve a simple variational technique. These theorems were motivated by applications to a problem in stochastic processes, where the matrices arise as covariance matrices. The applications are discussed in §2.

1. **The convexity conditions.** Let $n \geq 3$ be a fixed integer, I the $n \times n$ identity matrix, and E the $n \times n$ matrix with all entries 1. Let $(M)_{ij}$ denote the i, j-th entry of a matrix M. The convexity conditions which we discuss can most easily be stated in terms of the $n \times n$ "differencing" matrix

$$D = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

We shall say that a (real) $n \times n$ matrix Q satisfies the *Convexity Condition* C_1 if and only if the matrix QD has positive diagonal entries and negative off-diagonal entries.

EXAMPLE 1. The matrix with entries

$$q_{ij} = f\left(\frac{i-j}{n}\right) \qquad (i, j = 1, 2, \cdots n)$$

satisfies C_1 if the function $f(t)$ on $-1 \leq t \leq 1$ is even, and on $0 \leq t \leq 1$ is monotone-decreasing, and strictly convex

$$f(\tfrac{1}{2}t_1 + \tfrac{1}{2}t_2) < \tfrac{1}{2}f(t_1) + \tfrac{1}{2}f(t_2),\ t_1 \neq t_2.$$

THEOREM 1. *Let Q be a positive-definite matrix satisfying Condition C_1. Then Q^{-1} has positive row and column sums, i.e., the matrix P defined by $PQ = E$ has all entries positive.*

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