TWO COMBINATORIAL THEOREMS ON ARITHMETIC PROGRESSIONS

By Wolfgang M. Schmidt

1. **Introduction.** According to a well-known theorem of van der Waerden [6] there exists an $m(k, l)$ defined for integers $k \geq 2$, $l \geq 3$, such that if we split the integers between 1 and m into k classes, at least one class contains an arithmetic progression of l distinct elements. We shall prove

Theorem 1. *For some absolute constant* $c > 0$,

$$m(k, l) \geq k^{l-c(l \log l)^{\frac{1}{2}}}. \tag{1}$$

For large l this is an improvement of the estimate

$$m(k, l) \geq [2(l-1)k^{l-1}]^{\frac{1}{2}} \tag{2a}$$

given by Erdös and Rado [2] and of the estimate

$$m(k, l) \geq lk^{C \log k} \tag{2b}$$

of Moser [4].

Throughout, $P, Q, \cdots$ will denote arithmetic progressions of l distinct integers between 1 and m. Consider real numbers α between 0 and 1 written in scale $k : \alpha = 0, \alpha_1\alpha_2 \cdots$. Write $N(\alpha; k, l, m)$ for the number of progressions $P = \{p_1, \cdots, p_l\}$ such that

$$\alpha_{p_1} = \alpha_{p_2} = \cdots = \alpha_{p_l}.$$

Theorem 2. *Keep* $k, l, \epsilon > 0$ *fixed. Then for almost every* α,

$$N(\alpha; k, l, m) = m^2 \frac{k^{1-l}}{2(l-1)} + O(m \log^{\frac{3}{2}+\epsilon} m). \tag{3}$$

2. **The idea of the proof of Theorem** 1. There is a 1-1 correspondence between divisions of $1, \cdots, m$ into classes $C_1, \cdots, C_k$ and functions $f(x)$ defined on $1, \cdots, m$ whose values are integers between 1 and k. We write

$$f(\sigma) = j$$

for a set σ of integers between 1 and m if $f(x) = j$ for every $x \in \sigma$. Put

$$P \mid f$$

if $f(P)$ is defined, i.e., if $f(p_1) = \cdots = f(p_l)$ for the elements $p_1, \cdots, p_l$ of P. In this terminology Theorem 1 means that for $m < k^{l-c(l \log l)^{\frac{1}{2}}}$ there exists some f such that $P \mid f$ for no P.

Received May 19, 1961.