THEORY OF COMPACT RINGS. III. COMPACT DUAL RINGS

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1. **Introduction.** Let A be a topological ring and L a closed left ideal of A. The right annihilator $r(L)$ of L is a closed right ideal of A. "Dual rings" are topological rings such that the mapping $L \rightarrow r(L)$ sets up a one-to-one correspondence between the closed left ideals and the closed right ideals of A.

The notion of dual rings, introduced by Baer [1] and Kaplansky [7], is a natural generalization of that of quasi-Frobenius algebras which were first defined and studied by Nakayama [8], [9]. Really, discrete dual rings with identity and with minimum condition for one-sided ideals are quasi-Frobenius rings [9; Theorem 6].

In the present paper we shall study a certain analogy between compact dual rings and quasi-Frobenius rings.

In the paper [4], "compactness" means "Frechet's compactness", and the underlying topological spaces of division rings are assumed to be separable.

In the present paper "compactness" means "bicompactness in the sense of Alexandroff and Urysohn", and we do not assume separability of the spaces.

However, it is known that the same theory as that given by Jacobson [4] can be developed on non-discrete locally compact (= locally bicompact) totally disconnected division rings.

After elementary preparations in §§2 and 3, we shall obtain, in §4, a criterion for compact rings with identity to be dual rings ((a_1) and (a_2) in §4), from which we can get some properties of compact dual rings. In particular, it follows that compact dual rings with open radical are finite.

In §5, we shall consider Shoda's condition, and obtain, for compact rings with identity, a theorem similar to Ikeda's [2; Theorem 1].

In the last section, §6, we shall study compact rings in which closed ideals are principal ideals, and former results of the author [12; Theorems 5.2 and 5.3] will be treated from the standpoint of the theory of dual rings.

We shall use the term ring to denote topological ring. *Identity element* (or simply *identity*) of a ring (if it exists) will be denoted by 1. Unless otherwise stated, the word *ideal* will mean two-sided ideal and *radical* always means Perlis-Jacobson radical. We denote by $\bar{B}$ the topological closure of a subset B of a ring. Throughout the paper we shall use the terminology of Jacobson [5] and Kaplansky [6].

2. **Introductory notions.** If S is a non-vacuous subset of a ring A, we denote by $l(S)$ and $r(S)$ the left and right annihilators of S, respectively, i.e., $l(S) = \{x \mid x \in A, xS = 0\}$ and $r(S) = \{y \mid y \in A, Sy = 0\}$. It is evident that $l(S)[r(S)]$

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