SEPARABILITY OF CONNECTED, LOCALLY CONNECTED, METRIC SPACES

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Introduction. In [1] Alexandroff has shown that if a connected metric space S is locally separable, then S is separable. In [2] Jones has shown that if a connected, *locally connected*, metric space S is locally *peripherally* separable, then S is separable. Treybig in [5] gave an example of a connected, *semi-locally-connected*, metric space, which is locally peripherally separable, but which is not separable. This example affirms the distinction between the locally connected situation and the situation where local-connectedness is omitted. Recently Treybig in [6] and Roberts in [4] have obtained some results which complement Alexandroff's theorem. This paper develops an analogous idea in the direction of Jones' theorem in the locally connected situation. In this case it is shown that *if no point separates a connected, locally connected, metric space S, and each pair of points in S is separated by a separable closed set, then S is separable.*

Notation. Let S denote a connected, locally connected, metric space. If each of p and q is a point in S, then $d(p, q)$ denotes the distance between p and q. If p is a point in S and r is a positive number, then $D_r(p)$ denotes the open disc with center p and radius r.

Theorem. *If* (1) *no point separates S,* (2) *there is a separable closed set which separates S, and* (3) *each separable closed set which separates S contains two points which are separated by a separable closed set* (*this includes* 1), *then S is separable.*

Proof. Suppose that S satisfies the hypothesis of the theorem, but S is not separable. Let it be that A_0 is a separable closed set which separates S; B_0 is a set to which a point p belongs only in case p is in A_0 and A_0 contains a point q which is separated from p by a separable closed set; C_0 is a countable dense subset of B_0 ; E_0 and F_0 are functions such that for each point p in C_0 , $E_0(p)$ is a set to which q belongs only in case q is in A_0 and q is separated from p by a separable closed set, and $F_0(p)$ is a countable dense subset of $E_0(p)$; N_0 and G_0 are functions such that for each point p in C_0 , and each point q in $F_0(p)$, $N_0(p, q)$ is the least positive integer m such that $D_{1/m}(p)$ is separated from $D_{1/m}(q)$ by a separable closed set, and $G_0(p, q)$ is a separable closed set whch separates $D_{1/N_0(p,q)}(p)$ from $D_{1/N_0(p,q)}(q)$; and $H_0 = A_0 + \sum_{p\varepsilon C_0} \sum_{q\varepsilon F_0(p)} G_0(p, q)$.

Let $A_1 = \bar{H}_0$. We proceed by induction. If n is a non-negative integer, $\bar{H}_n$ is a separable closed set which contains two points which are separated by a separable closed set. Set $A_{n+1} = \bar{H}_n$, and let B_{n+1} , C_{n+1} , E_{n+1} , F_{n+1} , N_{n+1} ,

Received February 28, 1961. Presented to the American Mathematical Society January, 1961. The author is greatly indebted to Professor F. B. Jones for his guidance in the preparation of this paper.