ASYMPTOTIC RENEWAL THEOREMS IN THE ABSOLUTELY CONTINUOUS CASE

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1. **Introduction.** This paper presents some results on the solution $p(\cdot)$ of the integral equation of renewal theory

$$p(t) = \int_0^t p(x)g(t-x)\,dx + h(t). \tag{1}$$

We assume in the following that $g(\cdot)$ and $h(\cdot)$ are nonnegative Lebesgue measurable functions such that

$$1 = \int_0^\infty g(x)\,dx, \tag{2}$$

and

$$\text{if} \quad t > 0, \qquad \int_0^t h(x)\,dx < \infty. \tag{3}$$

The existence and uniqueness of solutions to (1) have been discussed by Feller [3]. For our purposes it is sufficient to know there is a uniquely determined function $p(\cdot)$ solving (1) and satisfying

$$\text{if} \quad t > 0, \qquad p(t) \geq 0 \quad \text{and} \quad \int_0^t p(x)\,dx < \infty. \tag{4}$$

Throughout, μ is defined by

$$\mu = \int_0^\infty xg(x)\,dx. \tag{5}$$

$\mu = \infty$ is allowed and if $\mu = \infty$, the value of $1/\mu$ is zero.

In §2 we prove the following three theorems using only standard measure theory results.

THEOREM 1. *Suppose $p(\cdot)$ is a bounded nonnegative solution of* (1).

(A) *If $\mu < \infty$, then $\int_0^\infty h(x)\,dx < \infty$.*
(B) *If $\lim_{x\to\infty} p(x)$ exists, then $\lim_{x\to\infty} h(x) = 0$.*
(C) *If $\mu < \infty$ and $\lim_{x\to\infty} h(x) = 0$, then*

$$\lim_{x\to\infty} p(x) = (1/\mu)\int_0^\infty h(x)\,dx.$$

(D) *If $\mu = \infty$, $\lim_{x\to\infty} h(x) = 0$ and $\int_0^\infty h(x)\,dx < \infty$, then $\lim_{x\to\infty} p(x) = 0$.*

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