MAHLER MATRICES AND THE EQUATION $QA = AQ^m$

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1. Introduction. In this article several sets of matrices are defined; each set forms an Abelian group. The elements of the matrices are 0, 1, -1 and other roots of unity (or sums of roots of unity). The determinant, characteristic roots, vectors, and elementary divisors are also found.

Thus the matrices form a convenient set of test matrices for a routine which purports to solve a matrix. A typical test matrix may involve nonreal elements; if each element of the matrix is replaced by a 2×2 matrix in the usual way:

$$a + bi \sim \begin{bmatrix} a, & b \\ -b, & a \end{bmatrix},$$

the roots of the expanded real matrix are the roots of the test matrix and their conjugates. The vectors of the expanded matrix are easily found also.

Indeed if $z = [z_1, z_2, \cdots, z_n]'$ is a (column) vector of the matrix $A: Az = z\lambda$, $z_n = x_n + iy_n$, the vector of the real expanded matrix which corresponds to the root λ is $[x_1 + iy_1, ix_1 - y_1, \cdots, x_n + iy_n, ix_n - y_n]'$. If λ is real, the vectors $[x_1, -y_1, \cdots, x_n, -y_n]'$, $[y_1, x_1, \cdots, y_n, x_n]'$ of the expanded matrix also correspond to λ [3].

In view of the group property and a certain isomorphism of the group, the product of several matrices from a set is easily written down at sight. This gives a convenient test of the corresponding computation routine.

A set of matrices was defined by K. Mahler [6] which also forms an Abelian group. His set is a special one of the sets defined here. The roots and determinants of Mahler's special matrices were given by Lehmer [5]. I am indebted to Lehmer for calling my attention to his and Mahler's work. The results of Lehmer and Mahler inspired this paper.

2. The matrices form an Abelian group. Let l, n be positive integers, $(l, n) = 1, l > 1, n > 1$. (The case $l = 1$ is taken up in §8.) Set $s = \exp [2\pi i/l]$, and let m be a positive integer, $(m, n) = 1$, $m \equiv 1 \pmod{l}$, i.e., $l \mid m - 1$.

Definition 1. Let Q be the matrix

$$\begin{bmatrix} 0, 1 & & & \\ 0, 0, 1 & & & \\ \cdot \quad \cdot \quad \cdot & & & \\ 0, 0, \cdots, 0, 1 & & & \\ s, 0, \cdots, 0, 0 & & & \end{bmatrix}.$$

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