THE LEXICOGRAPHIC PRODUCT OF GRAPHS

By Gert Sabidussi

1. **The automorphism group of the lexicographic product.** The purpose of this section is to improve an earlier result [3] giving a necessary and sufficient condition under which the automorphism group of the lexicographic product of two graphs X, Y is equal to the wreath product ([1; 81]; [3, Definition 1]) of the groups of X and Y. Using the terminology and notation of [3] we have the following:

Theorem 1. *Let X be any graph with $E(X) \neq \Box$, and suppose that Y is such that $| V(Y; y) \cap V(Y; y') | < | Y |$ for any two distinct vertices y, y' of Y. Then a necessary and sufficient condition that $G(X) \circ G(Y) = G(X \circ Y)$ is that Y be connected if $R \neq \Delta$, and that Y' be connected if $S \neq \Delta$.*

In [3], X and Y were assumed to be almost locally finite, and finite, respectively. Finite graphs trivially satisfy the condition that $| V(Y; y) \cap V(Y; y') | \leq d(Y; y) < | Y |$.

The proof of Theorem 1 will be broken up into a sequence of lemmas. Note first that if X and Y satisfy the hypotheses of the theorem, then $X \circ Y$ is not isomorphic to Y. For if it were, then $| V(X \circ Y; (x, y)) \cap V(X \circ Y; (x, y')) | < | Y |$ for any $x \in X$, and distinct vertices $y, y' \in Y$. But if $d(X; x) \geq 1$, then

$$| V(X \circ Y; (x, y)) \cap V(X \circ Y; (x, y')) | \geq | V(X; x) \times V(Y) | \geq | Y |.$$

It follows from this that if $| K_x | = 1$, then $\phi Y_x = Y_{x'}$ with $| K_{x'} | = 1$.

It therefore suffices to consider the case $| Y | \geq 2$, $| K_x | \geq 2$, and to prove that C_x is complete. Let $c_1, c_2 \in C_x$, $y_i \in B_{c_i}$, $i = 1, 2$, and suppose that $\rho_X(c_1, c_2) = 2$. *This assumption is made throughout the following sequence of lemmas.*

Lemma 1. *Let A, B, C, M, N be sets such that $| N | < | B |$, and $A \times B = (C \times B) \cup M - N$, then $A \supset C$. If, moreover, $| M | < | B |$, then $A = C$.*

This is obvious.

Lemma 2. *$V(X; t_y) \subset V(X; c_1) \cap V(X; c_2)$ for each $y \in Y$ with $d(Y; y) < | Y |$.*

Proof. Consider $W = V(X \circ Y; (x, y_1)) \cap V(X \circ Y; (x, y_2))$. Since $\rho_X(c_1, c_2) = 2$,

$$\phi W = (V(X; c_1) \cap V(X; c_2)) \times V(Y) \neq \Box. \tag{1}$$

For W itself we obtain $W = (V(X; x) \times V(Y)) \cup D$, where

$$D = \{x\} \times (V(Y; y_1) \cap V(Y; y_2)).$$

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