UNITARY FUNCTIONS (mod r)

In memoriam, Arthur N. Milgram, 1912–1961.

BY ECKFORD COHEN

1. Introduction. If r is a natural number, then a positive divisor d of r such that $d\,\delta = r$, $(d, \delta) = 1$, is called a *unitary divisor* of r. For integers n, we denote by $(n, r)_*$ the largest divisor of n which is a unitary divisor of r. In case $(n, r)_* = 1$, n will be called a semiprime (unitarily prime) to r.

Suppose now that $f(n, r)$ is a complex-valued function defined for all n such that $f(n, r) = f((n, r)_*, r)$. Then we call $f(n, r)$ a *unitary function* of n (mod r). Clearly, the unitary functions (mod r) form a subclass of the even functions (mod r); we recall that $F(n, r)$ is even (mod r) if $F(n, r) = F((n, r), r)$ for all n.

In this paper we develop arithmetical and trigonometric inversion theories for the class of unitary functions (mod r). The results obtained are analogous to existing results for the more general class of even functions (mod r). It is convenient at this point to recall the inversion formulas for the latter class of functions.

Suppose that $F(n, r)$ is even (mod r) and $r = r_1 r_2$. Further, let $c(n, r)$ denote Ramanujan's trigonometric sum, $\phi(r) = c(0, r)$, $\mu(r) = c(1, r)$. Then by [3, Theorem 2.1]

$$(1.1)\qquad F(n, r) = \sum_{d \mid (n,r)} G\left(d, \frac{r}{d}\right) \rightleftarrows G(r_1, r_2) = \sum_{d \mid r_1} F\left(\frac{r_1}{d}, r\right)\mu(d);$$

moreover, it was shown in [2, Theorem 1] that

$$(1.2)\qquad F(n, r) = \sum_{d \mid r} \alpha(d, r) c(n, d) \rightleftarrows \alpha(d, r) = \frac{1}{r\phi(d)} \sum_{m \,(\mathrm{mod}\, r)} F(m, r) c(m, d).$$

In case $F(n, r)$ is defined by the left of (1.1), then [2, (10)] the Fourier coefficients $\alpha(d, r)$ are determined by

$$(1.3)\qquad \alpha(d, r) = \frac{1}{r} \sum_{e \mid r/d} G\left(\frac{r}{e}, e\right) e.$$

The relation (1.1) was proved by purely arithmetical means, while (1.2) was shown to result from the orthogonality property [1, (3.10)],

$$(1.4)\qquad \sum_{n \equiv a+b \,(\mathrm{mod}\, r)} c(a, d_1) c(b, d_2) = \begin{cases} rc(n, d) & \text{if } d_1 = d_2 \equiv d, \\ 0 & \text{if } d_1 \neq d_2, \end{cases}$$

where d_1, d_2 are divisors of r, the summation being over all a, b (mod r) such that $n \equiv a + b$ (mod r).

Received February 27, 1961.