UNIFORM APPROXIMATION BY POLYNOMIALS WITH POSITIVE COEFFICIENTS

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In this paper we discuss special cases of the following problem. Let A be a compact Hausdorff space and $C^+(A)$ be the cone of all positive continuous functions on A. (Throughout this paper, A is to contain at least two points. Terms such as "positive", "increasing", etc. are understood in the wide sense, not excluding the case of equality.) Under what conditions for a given set $G \subset C^+(A)$ is it true that each $f \in C^+(A)$ can be uniformly approximated by linear combinations of functions in G with positive coefficients or by polynomials in functions from G with positive coefficients? Briefly, when are the "positive linear combinations" or the "positive polynomials" dense in $C^+(A)$? Typical examples are provided by the theorems that the positive linear combinations of $x^m y^n (1 - x - y)^k$ for $m, n, k = 0, 1, \cdots$ are dense in $C^+(A)$, if A is the triangle $x \geq 0, y \geq 0, x + y \leq 1$ [2; 51] or that those of $e^{-nx}x^k$ are dense in $C^+ [0, +\infty]$ restricted to functions which vanish at infinity. There is a considerable difference in approximation by positive linear combinations compared with arbitrary linear combinations. The density of the positive linear combinations seems to require that there exist functions in G with an arbitrarily steep maximum at an almost arbitrary prescribed point x of A. The density of positive polynomials seems to result from high powers of functions in G with a strict maximum at x. By contrast, the density of arbitrary polynomials depends upon separation properties of G (the Weierstrass-Stone theorem), and the density of arbitrary linear combinations represents a problem hardly touched in generality (closure theorems).

Approximation by positive linear combinations (§2) and by positive polynomials in two functions (§3) are first discussed for subsets A of the real line. This leads to a complete solution of the polynomial problem for arbitrary A in case when there are only two functions in G. For n-dimensional sets A (§4; we treat only the two-dimensional case which is typical) new difficulties appear. They have their origin in the additional assumptions of the regularity of the shape of the covering sets which are necessary for the validity of the covering theorems of Vitali type. But also here we are able to solve the problem for the most natural cases. The question whether the necessary conditions 1.3 (§1) are sufficient in general, remains open.

1. **General necessary and sufficient conditions.** It is convenient to consider a slightly more general problem. Let A be a compact Hausdroff space, B a

Received June 29, 1960. This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract No. AF 49 (638)-619.