A SET OF SQUARE-WAVE FUNCTIONS ORTHOGONAL AND COMPLETE IN $L_2(0, 2)$

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1. **Introduction.** We define

$$(1)\qquad \left.\begin{aligned} S_1(x) &= 1, &\quad 0 < x < 1,\\ &= -1, &\quad 1 < x < 2,\end{aligned}\right\}$$

$$S_1(0) = S_1(1) = 0,$$

$$\text{and}\quad S_1(x+2) = S(x).$$

Note that $S_1(x) = \operatorname{sgn}(\sin \pi x)$.

This is evidently an odd function. We define the related even function

$$(2)\qquad C_1(x) = S_1(x + 1/2) = \operatorname{sgn}(\cos \pi x).$$

The functions $S_1(2^n x)$ for $n = 1, 2, \cdots$ are known as Rademacher functions [7], [5], [12]. It is elementary to prove that these functions are orthogonal on (0, 1), but not complete.

From the Rademacher functions one may construct the set of Walsh functions [10], [4], [6], [9]. For each positive integer m expressed in binary representation, namely

$$m = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_k},$$

write

$$\psi_m(x) = S_1(2^{m_1+1}x)S_1(2^{m_2+1}x) \cdots S_1(2^{m_k+1}x).$$

This set of functions is orthogonal and complete; it is used extensively in probability and in statistics [2], [3], [11].

In this paper, we define two sequences of functions, $\{S_n(x)\}$ and $\{C_n(x)\}$, consisting of linear combinations of $S_1(kx)$ and $C_1(kx)$ respectively. Each of these systems of functions is orthogonal on (0, 1) and complete in $L_2(0, 1)$, and together they form a complete orthogonal system in $L_2(0, 2)$.

2. **The sequences of functions $\{S_1(nx)\}$ and $\{C_1(nx)\}$.** A theorem established by Szász [8] may be stated as follows.

Theorem 1. (Szász) *Let $\phi(x)$ be bounded, $-\infty < x < \infty$, and such that the sequence $\{\phi(nx)\}$, $n = 1, 2, \cdots$, is orthogonal on (a, b) and complete in $L_2(a, b)$. If $\psi(x) \in L_2$ and if its Fourier coefficients a_n (with respect to $\{\phi(nx)\}$) satisfy the*

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