THE KERNEL OF A SEMIGROUP OF MEASURES

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In a previous paper [3, Theorem 3] it was shown that the kernel (= minimal ideal) of a compact affine topological semigroup consists of idempotents (see Lemma 1 below). In this work we consider some of the implications of this theorem when applied to a particular affine semigroup, namely, the convolution semigroup of (non-negative normalized regular Borel) measures on a given compact topological semigroup S. The notation $S^{\sim}$ will be adopted throughout to denote this measure semigroup, while $K^{\sim}$ will denote its kernel. The main result of this paper is a characterization (obtained partially by use of Lemma 1) of $K^{\sim}$, in terms of certain maximal groups of S. The fundamental Lemma 1 is also used here to produce, when S is a group, an idempotent whose carrier is all of S. From this it follows rather easily (using also Lemma 2) that S has Haar measure when S is a group. Finally, several corollaries to the main result are given, among them the theorem that $K^{\sim}$ is convex if and only if S has either a right invariant or a left invariant mean.

For terminology and notation, the papers numbered [9], [4], and [8] are suggested. Because of their repeated use in the proofs which follow, we first state (as Lemmas 1 and 2) and prove two theorems which have appeared elsewhere (see [3, Theorem 3] and [2, Theorem 2]). This first lemma was first proved by A. L. Shields (unpublished).

Lemma 1. *The kernel of any compact affine topological semigroup T consists of idempotents.*

Proof. Let G be any maximal group of T contained in the kernel of T (the kernel is the union of such groups [8, Theorem 2.2]). Since $G = eTe$ for some $e^2 = e$ and the mapping $x \rightarrow e\,x\,e$ is affine and continuous from T onto G, G is convex and compact. A simple application of the separation theorem (e.g., see [5; 361]) shows that for fixed x in G the mapping $y \rightarrow xy$ has a fixed point. Thus, there exists y in G for which $y = xy$, so $x = e$. This completes the proof.

Denote by $C(S)$ the space of complex continuous functions on S, and if $\mu \in S^{\sim}$, let $f'(x) = \mu(f^x)$, where $f^x(y) = f(yx)$. It is known that both f^x and f' belong to $C(S)$ [4; 53]. It should be remarked the notation f' introduced above depends on both f and μ. However, when it is used in the following, the measure μ is always fixed, so no ambiguity can result. A fact of use in the following lemma is that the convolution product $\mu\nu$ of two measures μ and ν is defined in such a way that $(\mu\nu)(f) = \nu(f')$.

We say the compact semigroup H of S is *simple* if the kernel of H is all of H.

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