THE SPACES L^P, WITH MIXED NORM

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1. Introduction. In this paper we shall deal with spaces of measurable functions defined in the following way. Let (X_i, S_i, μ_i), for $1 \leq i \leq n$, be n given, totally σ-finite measure spaces and $P = (p_1, p_2, \cdots, p_n)$ a given n-tuple with $1 \leq p_i \leq \infty$. We always suppose that none of the spaces (X_i, S_i, μ_i) admits as the only measurable functions the constant ones. A function $f(x_1, \cdots, x_n)$ measurable in the product space $(X, S, \mu) = (\prod_{i=1}^n X_i, \prod_{i=1}^n S_i, \prod_{i=1}^n \mu_i)$ is said to belong to $L^P(X)$ if the number obtained after taking successively the p_1-norm in x_1, the p_2-norm in x_2, $\cdots$, the p_n-norm in x_n, and in that order, is finite. The number so obtained, finite or not, will be denoted by $\| f \|_P$, $\| f \|_{(p_1, \cdots, p_n)}$ or $\| f \|_{p_1, \cdots, p_n}$. When for every i, $p_i < \infty$, we have in particular:

$$\| f \|_P = \left(\int \cdots \left(\int \left(\int | f(x_1, \cdots, x_n) |^{p_1} d\mu_1\right)^{p_2/p_1} d\mu_2\right)^{p_3/p_2} \cdots d\mu_n\right)^{1/p_n}.$$

If further, each p_i is equal to p:

$$\| f \|_P = \| f \|_{(p, \cdots, p)} = \left(\int | f(x_1, \cdots, x_n) |^p d\mu\right)^{1/p}$$

$$= \| f \|_p \text{ and } L^P(X) = L^p(X).$$

The paper is divided in two parts. Part I contains general properties of these spaces and a generalization of well-known interpolation theorems. Part II deals with the case when every component space (X_i, S_i, μ_i), is Euclidean with Lebesgue's measure, and there are studied properties of continuity of some well-known operators. We shall return to this last point in another paper. With few exceptions each result mentioned, known or not, is accompanied by a proof. For notations, nomenclature and general references we refer to [8], [14] and [15].

Finally, we wish to mention that the L^P spaces with mixed norm are a particular case of the so-called, Banach function spaces, (cf. [11]). Using the results of the Banach function spaces some sections of the present paper could be shortened.

PART I.

2. Basic results. Throughout the paper, the letters $P, Q, R, \ldots$, will designate n-tuples ($n \geq 1$), $P = (p_1, \cdots, p_n), \cdots, R = (r_1, \cdots, r_n)$ with com-

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