INTRINSIC FUNCTIONS ON MATRICES

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1. **Introduction.** Let $\mathfrak{A}$ be a linear associative algebra, with identity, over a field $\mathfrak{F}$, and let F be a single-valued function with domain, $\mathfrak{D}$, and range in $\mathfrak{A}$. Let $\mathfrak{G}$ be the group of all automorphisms and antiautomorphisms of $\mathfrak{A}$ which leave $\mathfrak{F}$ elementwise invariant.

Definition 1.1. *The function F is called intrinsic, if for any $\Omega \in \mathfrak{G}$:*

(1) *$Z \in \mathfrak{D}$ implies $\Omega Z \in \mathfrak{D}$*
(2) *$Z \in \mathfrak{D}$ implies $f(\Omega Z) = \Omega f(Z)$.*

In other words an intrinsic function is one which admits the group $\mathfrak{G}$ as operator domain.

The concept of intrinsic functions was motivated and introduced in [9], and studied for finite-dimensional algebras $\mathfrak{A}$ over the real and complex fields. It was shown that *primary functions*, i.e. those functions arising from ordinary functions of a complex variable, according to the well-known extension of such functions to linear algebras [8] are always intrinsic. For the algebra of real quaternions it was demonstrated that all intrinsic functions are primary, thereby establishing a complete characterization of intrinsic functions on quaternions.

The goal of the present paper is the characterization of the intrinsic functions on total matrix algebras $\mathfrak{M}_C^n$ of $n \times n$ matrices over the complex field $\mathfrak{C}$. Essentially complete characterization is achieved for appropriately continuous functions on $\mathfrak{M}_C^n$. A subclass of these functions yields a class of intrinsic functions on $\mathfrak{M}_R^n$, the algebra of real matrices.

Since the group $\mathfrak{G}$ for $\mathfrak{M}_C^n$ includes the inner automorphisms, i.e. those given by $T^{-1}\mathfrak{M}_C^n T$ where T is a non-singular complex matrix, the characterization of intrinsic functions on $\mathfrak{M}_C^n$ provides a characterization of general functions of linear transformations on an n-dimensional complex vector space.

It turns out that the general intrinsic function for matrices of order n over the complex field is of the form $f(Z, \sigma_1[Z], \cdots, \sigma_{n-1}[Z])$ where the $\sigma_i[Z]$ are the first $n - 1$ elementary symmetric functions of the eigenvalues of the argument matrix Z, and where $f(Z, \sigma_1[Z], \cdots, \sigma_{n-1}[Z])$ is obtained from the scalar function $f(z, \sigma_1, \cdots, \sigma_{n-1})$ of the complex variable z and the $n - 1$ complex parameters $\sigma_1, \cdots, \sigma_{n-1}$ in a manner quite analagous to the classical mode of generalization of a function of a single complex variable to a matrix algebra.

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