THE q-STIRLING NUMBERS OF FIRST AND SECOND KINDS

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1. **Introduction.** In a recent paper Carlitz [3] has considered the polynomial

$$\beta_k^{(z)}(x, q) = (q - 1)^{-k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left(\frac{j+1}{[j+1]}\right)^z q^{jx} \tag{1.1}$$

where

$$[j] = (q^j - 1)/(q - 1)$$

and z is any real number as a generalization for q-series of the Bernoulli polynomials of Nörlund [8]. Sharma [9] approached the extension in a different manner and defined

$$\bar{\beta}_k^{(n)}(x, q) = (q - 1)^{-k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(j+n)_n}{[j+n]_n} q^{jx} \tag{1.2}$$

where

$$(j + n)_n = (j + 1) \cdots (j + n), \qquad [j + n]_n = [j + 1] \cdots [j + n],$$

with n assumed to be a positive integer.

Seeking to define a suitable polynomial analogue for negative values of n, Carlitz [3] then gave the definition

$$\bar{\beta}_k^{(-n)}(x, q) = (q - 1)^{-k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{[j+n]_n}{(j+n)_n} q^{jx}, \tag{1.3}$$

where n is a positive integer.

Since there is evidently no unique method in q-series theory for extending formulae common for ordinary numbers, it may be of interest to exhibit a development of these matters by the device of starting with a combinatorial definition of what we shall mean by q-Stirling numbers which preserves the usual Stirling number relations and leads perhaps more naturally to a desired q-Bernoulli polynomial, or q-Stirling polynomial, and which is really an extension of (1.3) avoiding (1.1) and (1.2).

The reason for attempting this is to avoid the device of interchanging parentheses and square brackets in going from (1.2) to (1.3). Presumably a parallel theory could be made to rest on (1.2); however, we shall not discuss this here. In the development which follows, we note particularly relations (3.10), (3.11), (3.18), (3.19), (3.21), and (3.22) which appear to be either new or else new ways of expressing older ideas more comprehensively.

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