GENERALIZED KUMMER CONGRUENCES FOR THE PRODUCTS OF SEQUENCES. APPLICATIONS

By Harlan Stevens

1. **Introduction.** In attempting to prove a result involving Bernoulli numbers, Kummer [12] was able to show that certain sequences of rational numbers satisfy quite general congruences. In particular, he proved that if certain sufficient conditions hold, then a sequence $\{A_n\}$ will satisfy

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} A_{n+s(p-1)} \equiv (\text{mod } p^r) \tag{1.1}$$

for all $n \geq r \geq 1$ and where p is a rational prime. Since then, other writers have considered "Kummer's congruences". Various methods have been employed, for example, to show that the Euler numbers satisfy (1.1) for $p > 2$. Carlitz [7] has shown that the Hurwitz product of two sequences satisfying congruences similar to (1.1) also satisfies a like congruence. In a recent paper [16], using a generalized version of (1.1), we have extended these results to the Hurwitz product of an arbitrary number of sequences of either rational or algebraic numbers.

In this paper we will apply some of these general results to various familiar sequences, thereby obtaining more general sequences that satisfy Kummer's congruence. It is remarked by Nielsen [13, Chapter 14] that for $p = 2$ the proofs of (1.1) for the Euler numbers do not hold. Consequently we treat this situation in §5 and obtain Kummer's congruences (mod 2^r) for higher order Euler numbers. In particular, our results imply

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} k^{r-s} E_{n+2s}^{(k)} \equiv 0 \pmod{2^r},$$

where $n \geq 0$, $r \geq 0$ and $E_n^{(k)}$ is determined by

$$(\text{sech } x)^k = \sum_{n=0}^{\infty} E_n^{(k)} \frac{x^n}{n!}.$$

Still another situation is represented by the Bernoulli numbers. They also do not satisfy a congruence exactly like (1.1), and we take up these numbers in §7. In the main, the only property of our sequences that we shall use is that they satisfy congruences similar to (1.1).

2. **Preliminaries.** In every instance p will represent a fixed rational prime and R_p the set of rational numbers which are integral (mod p). If $\mathfrak{p}$ is any fixed prime ideal of $R(\eta)$, where $R(\eta)$ represents the algebraic number field

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