LOCAL CONNECTEDNESS OF INVERSE LIMIT SPACES

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1. **Introduction.** Let X_i be a metrizable continuum for each positive integer i, and let f_i be a mapping of X_{i+1} onto X_i. We let X be the inverse limit of the system $X_1 \overset{f_1}{\leftarrow} X_2 \overset{f_2}{\leftarrow} X_3 \overset{f_3}{\leftarrow} \cdots$, (written $X = \lim(\{X_i\}, \{f_i\})$), and we let π_i be the projection mapping of X onto X_i.

If X_i is the unit circle in the complex plane for each i, and $f_i(z) = z^2$ for all i and all z, then X is the dyadic solenoid. This example illustrates the fact that X may fail to be locally connected even though each of the coordinate spaces X_i is locally connected. Indeed, if the circles X_i are all given the usual metric, then the set $\{X_i \mid i \text{ a positive integer}\}$ of spaces is equi-uniformly locally connected.

It is easily seen, however, that whether or not a collection of metrizable spaces is equi-uniformly locally connected depends upon a particular assignment of metrics to the spaces. We show that if metrics d_i for the spaces X_i are chosen so as to satisfy a certain "admissibility condition", then X is locally connected if and only if the collection $\{(X_i, d_i) \mid i \text{ a positive integer}\}$ of metric spaces is equi-uniformly locally connected.

Next we show that it is possible to embed X and the X_i in the Cartesian product of the X_i in such a way that X is locally connected if and only if the sequence $X_1, X_2, X_3, \cdots$ converges 0-regularly to X. This result is then combined with known results about 0-regular convergence to obtain information about the inverse limits of spaces of certain special types. For example, if each X_i is a 2-sphere and X is locally connected and 2-dimensional, then X is a 2-sphere.

2. **Admissible sequences of metrics.** Let d_i be a metric for X_i for each positive integer i. The sequence $d_1, d_2, d_3, \cdots$ is *admissible* if there exists a metric d for X such that

$$\lim_{i\to\infty} d_i(\pi_i(u), \pi_i(v)) = d(u, v)$$

uniformly on $X \times X$.

THEOREM 1. *There exists an admissible sequence of metrics.*

Proof. Let D_i be a metric for X_i. We assume that D_i is chosen so that $D_i(x, y) \leq 1$ for all x and y in X_i. If $i > j$, we define f_{ij} to be the composite mapping $f_j \cdots f_{i-2}f_{i-1}$ from X_i onto X_j, and we define f_{ii} to be the identity mapping on X_i. We then define

$$d_i(x, y) = \sum_{j=1}^{i} 2^{-j} D_j(f_{ij}(x), f_{ij}(y))$$

Received August 19, 1960. This research was supported by a National Science Foundation grant NSF-G12972. M. K. Fort, Jr. is an Alfred P. Sloan Research Fellow.