IRREDUCIBILITY OF CERTAIN CLASSES OF LEGENDRE POLYNOMIALS

By R. F. McCoart

Introduction. The Legendre polynomial of degree n is the polynomial of the form

$$P_n(x) = \frac{1}{2^n} \sum_{\nu=0}^{m} (-1)^\nu \binom{n}{\nu} \binom{2n - 2\nu}{n - 2\nu} x^{n-2\nu},$$

where $m = [n/2]$. For odd n this polynomial has the trivial factor x. Dividing in this case by x, we define

$$L_n(x) = \frac{1}{2^n} \sum_{\nu=0}^{m} (-1)^\nu \binom{n}{\nu} \binom{2n - 2\nu}{n - 2\nu} x^{2m-2\nu}.$$

For nearly 50 years it has been conjectured that L_n is irreducible over the field of rational numbers, but it has been proved only for values of n whose p-adic representations have certain special forms. In the following let p denote an odd prime.

In 1912 J. B. Holt [2] proved L_n irreducible if

$$2^a \leq n \leq 2^a + 1,$$

$$p - 2 \leq n \leq p + 1,$$

$$2p - 2 \leq n \leq 2p - 1.$$

In 1913 he extended his results [3] to include

$$n = p - 4, \qquad p + 3, \qquad 2p - 4.$$

In 1924 H. Ille [4] showed that L_n is irreducible for

$$n = p - 3, \qquad p + 2, \qquad 2p - 3$$

and for $n = (p - 1)\, p^k + j$, where $j = 0, \pm 1, \pm 2, \pm 3, -4$. In 1951 J. H. Wahab proved that L_n is irreducible for [6] $n = K \cdot 2^a + j$, where $j = 0, 1, 2, 3$ and $K < 19$, and for $n = p + j$, where $j = 4, \pm 5, \pm 6, \pm 7, \pm 8$, and $n = 2p - j$, where $j = 5, 6, 7$, and 8 and where in each of these last two forms p must belong to certain residue classes, modulo 8. In 1956 Melnikov [5] proved irreducibility in the case of $n = (p + 1)\, p^k + j$, where $j = 0$ or 1, and in 1960 Wahab [7] in a second paper proved the conjecture for $n = (p - 1)(p^{a+1} + p^a)$ if $p < 17$, for $n = (p - 1)(p^{a+2} + p^a)$ if $p < 5$, and for $n = (p - 1)(p^{a+2} + p^{a+1} + p^a)$ if $p < 5$.

Holt stated that his results proved the conjecture for all values of n less

Received August 4, 1960.