ORTHONORMAL SERIES AND DENSITY OF INTEGERS

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1. Menchoff and Rademacher have independently proved the following theorem (cf. [3; 162]) *given* $\{\phi_n\}$ *an orthonormal system of functions on* [0, 1] *and a sequence of coefficients* $\{a_n\}$ *such that* $\sum_{n=2}^{\infty} (\log n)^2 \mid a_n \mid^2 < \infty$, *then the series* $\sum_{n=1}^{\infty} a_n \phi_n(t)$ *converges almost everywhere.* Furthermore, as Menchoff proved by example, the theorem is best possible in the sense that if $\omega(n)$ is a positive function of n such that $\omega(n) = o(\log^2 n)$, there is an orthonormal system $\{\phi_n\}$ and a sequence of reals $\{a_n\}$ such that $\sum_{n=1}^{\infty} \omega(n) \mid a_n \mid^2 < \infty$ but such that $\sum_{n=1}^{\infty} a_n\phi_n(t)$ diverges almost everywhere. No restrictions as to completeness or uniform boundedness are imposed.

The purpose of this note is to explore the consequences, in terms of convergence of orthonormal series, of hypotheses on the coefficient sequence of the type $\sum_{n=2}^{\infty} (\log n)^{\beta} \mid a_n \mid^2 < \infty$, $0 \leq \beta < 2$. In case $\beta = 1$, we show in §§2 and 3 that this implies the existence of a sequence $\{m_\mu\}$ of distinct, positive integers of upper density one such that the partial sums of the series $\sum_{n=1}^{\infty} a_n\phi_n(t)$ of index m_μ converge almost everywhere. This result is shown to be best possible in a logarithmic scale although it lacks the complete precision of the above theorem. We shall also have something to say in §4 about other values of β. The results are in general analogues of known theorems for ordinary Fourier series (cf. [1], [2]), in which case much more of a positive nature can be said.

2. By a lacunary sequence $\{n_k\}$ of positive integers, we mean as usual that there exists $\lambda > 1$ such that $n_{k+1}/n_k \geq \lambda$ for all k. Given a sequence $\{m_\mu\}$ of distinct positive integers, let $\sigma(n)$ be the number of m_μ not exceeding n. The sequence has a density $\lim \sigma(n)/n$ if this limit exists. Otherwise we may speak of the upper density $\limsup \sigma(n)/n$ and the lower density $\liminf \sigma(n)/n$. It has been shown for ordinary Fourier series that if f belongs to L^2, there is a sequence $\{m_\mu\}$ of upper density one such that $s_{m_\mu}(t; f)$ converges almost everywhere. (cf. [1; 396]). $s_m(t; f)$ denotes as usual the m-th partial sum of the Fourier series. The sequence $\{m_\mu\}$ depends on the function f but not the point t. It was stated without proof in [2; 299] that this is no longer true for general orthonormal series. Our first theorem includes this statement and gives what is very close to a weakest condition on the coefficients which will insure the existence of a sequence $\{m_\mu\}$ of upper density one such that

$$s_{m_\mu}(t) = \sum_{n=1}^{m_\mu} a_n\phi_n(t)$$

converges almost everywhere.

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