ON PROPERTIES OF ANALYTIC FUNCTIONS

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It will be shown that if f is a function of a complex variable, the existence of the first derivative of f implies 1) the existence of all derivatives; and 2) a power series expansion converging to f. The removable singularity theorem will be proved as will the fact that bounded entire functions are constant. Finally it will be shown that the uniform limit of differentiable functions is differentiable. There is no mention of integration and, in fact, this paper is self-contained except for one fact taken from topological analysis—namely that if f is differentiable and non-constant, then f is open. Furthermore, this fact is used only to make the following observation: if f is differentiable, then not only does the maximum modulus theorem apply to f, but also to the function

$$\frac{f(z_0) - f(z)}{z_0 - z}.$$

From this observation the results follow in an elementary way.

It is to be understood that all regions and open sets lie in the complex plane and that all functions considered are from subsets of the complex plane into the complex plane. The term open map (called strongly open [3]) is a function which maps open sets of the plane into open sets of the plane. If S is a circle, $I(S)$ denotes the interior of S. If f is differentiable at z_0, the function

$$\frac{f(z) - f(z_0)}{z - z_0}$$

represents the function

$$h(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

when $z \neq z_0$ and $h(z) = f'(z_0)$ when $z = z_0$. The function h will, of course, be continuous at z_0. A region is a connected open set.

Lemma 1. *Suppose* 0 *is a bounded open set and* f *is continuous on* $\bar{0}$ *and open on* 0. *Then if* W *is a complementary domain (component of the complement) of* $f(\bar{0} - 0)$, $f(0) \cap W \neq \phi$ *implies* $f(0) \supset W$.

Proof. Suppose $f(0) \cap W \neq \phi$. Then $f(0) \cap W$ is open in W and $f(0) \cap W = f(\bar{0}) \cap W$ is closed in W. Since W is connected, $f(0) \supset W$.

Lemma 2. *Suppose* V *is open and* $p \in V$. *If* f *is continuous on* V *and open on* $V - p$, *then* f *is open on* V.

Received June 24, 1960.