QUASI-CONTINUOUS FUNCTIONS ON PRODUCT SPACES

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1. Introduction. Following Kempisty [2], we say a function on a topological space X to a space Y is quasi-continuous (qc) at a point x of its domain if for every neighborhood U of x and for every neighborhood W of $f(x)$, there is a non-null open set in $U \cap f^{-1}(W)$. If a function is quasi-continuous at each point of its domain, it will be called quasi-continuous. These functions were used by Kempisty [2] to extend some results of Hahn and Baire on real valued functions of several real variable which are continuous in each variable separately. In this note we show that some of the results of Kempisty hold for more general spaces. We also obtain, Theorem 2, a condition in terms of quasi-continuity that a function be continuous.

S. Marcus [3] has shown that Bledsoe's neighborly functions [1] are equivalent to quasi-continuous functions and that if f is a derivative function which is continuous almost everywhere, then it is quasi-continuous. For some other results on neighborly functions see [5] and [4].

Let X, Y and Z be topological spaces, and f a function on $X \times Y$ to Z. For $p \in X$, f_p will denote the function on Y to Z defined by $f_p(y) = f(p, y)$. Similarly for $p \in Y$. A space will be called a Baire space if its non-null open sets are second category. A space will be called quasi-regular if for every non-null open set U there is a non-null open set V such that $\bar{V} \subset U$.

If f is qc and has its range in a metric space, one may show that the set of points where the oscillation of f is not less than $1/n$, n a positive integer, is nowhere dense and it follows that the points of discontinuity of f form a set of the first category.

2. Functions qc **in each variable separately.**

THEOREM 1. *Let X be a Baire space, Y be second countable, and M be a metric space with metric ρ. If f is a function on $X \times Y$ to M such that f_p and f_q are* qc *for all $p \in X$ and $q \in Y$, then f is* qc.

Proof. Suppose there is a point $(p, q) \in X \times Y$ such that f is not qc at (p, q). Then there is an $\epsilon > 0$ and a neighborhood $U \times V$ of (p, q) such that for every non-null open set $E \subset U \times V$ there is a point $(x, y) \in E$ such that $\rho(f(x, y), f(p, q)) \geq \epsilon$.

Since f_q is qc at p, there is a non-null open set $W \subset U$ such that for all $x \in W$, $\rho(f(x, q), f(p, q)) < \epsilon/3$.

Let $\mathcal{V}$ be a countable base for Y and let $\{V_n : n = 1, 2, \cdots\}$ be those elements

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